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SHANGHAI JIAO TONG UNIVERSITY

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Discrete-Time Fourier Transform

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LTI Systems

$$x_k[n] \to y_k[n] \implies \sum_k a_k x_k[n] \to \sum_k a_k y_k[n]$$

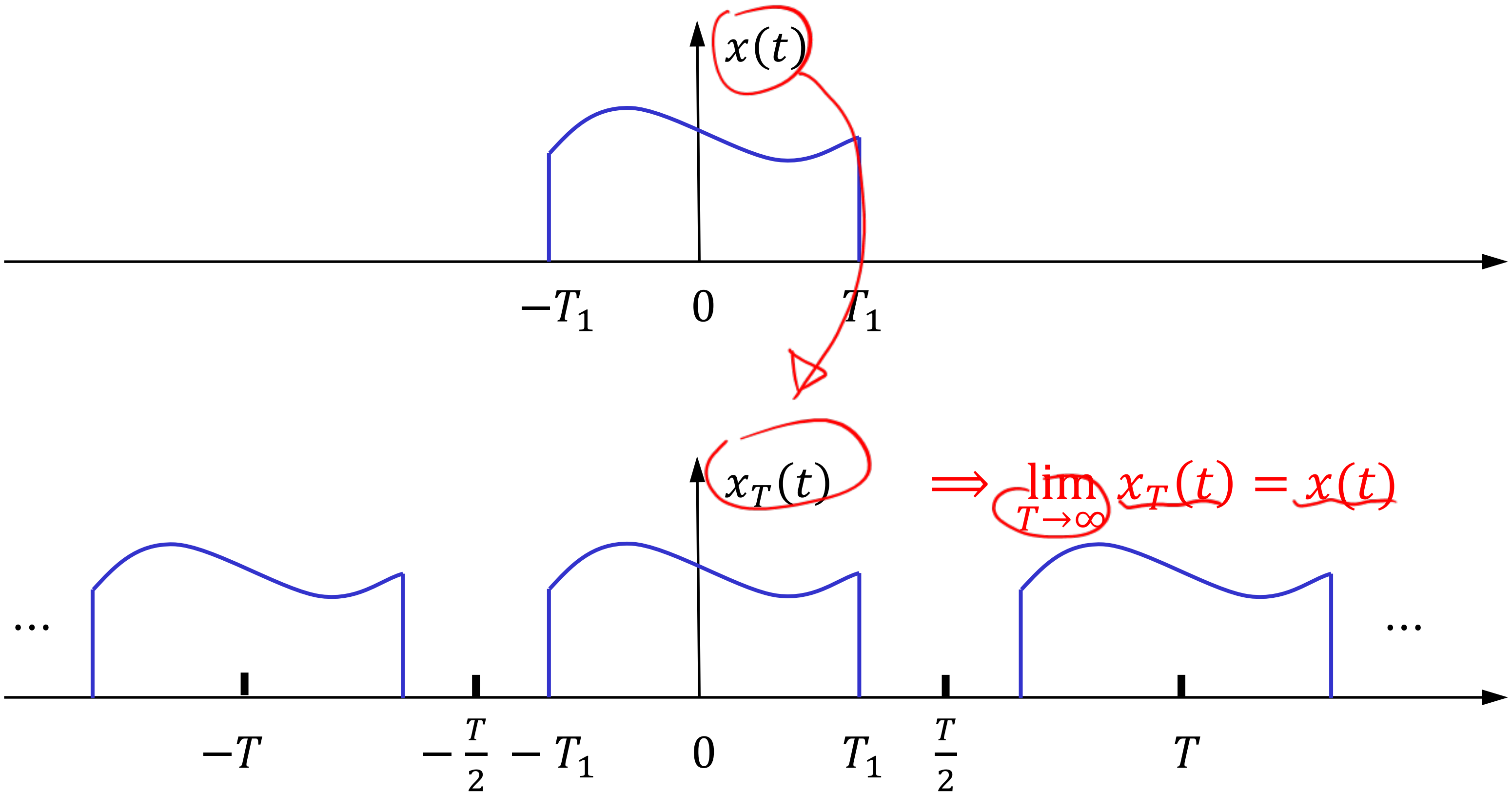
- **If we can find a set of “basic” signals, such that**
  - a rich class of signals can be **represented as linear combinations** of these basic (building block) signals.
  - the **response** of LTI Systems to these basic signals are both **simple and insightful**

$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$ $\quad H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$

- **Candidate sets of “basic” signals**
  - Unit impulse function and its delays: $\delta(t)/\delta[n]$
  - Complex exponential/sinusoid signals: $e^{st}, e^{j\omega t}/z^n, e^{j\omega n}$

CT Fourier Transform of Aperiodic Signal $x(t)$

- **General strategy**
  - approximate $x(t)$ by a periodic signal $x_T(t)$ with infinite period T



CT Fourier Transform Pair

- **Fourier transform (analysis equation)**

$$X(j\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

where $X(j\omega)$ called **spectrum** of $x(t)$

- **Inverse Fourier transform (synthesis equation)**

$$x(t) = \mathcal{F}^{-1}\{X(j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Superposition of complex exponentials at **continuum** of frequencies, frequency component $e^{j\omega t}$ has "amplitude" of $X(j\omega)d\omega/2\pi$

Fourier Series of DT Periodic Signals

- Recall a **periodic** DT signal $x[n]$
  - with fundamental period N, and
  - fundamental frequency $\omega_0 = 2\pi/N$
- **Fourier series** represent $x[n]$ in terms of harmonically related complex exponentials
  - **Synthesis equation**

$$x[n] = \sum_{k \in \langle N \rangle} a_k\, e^{jk\omega_0 n} = \sum_{k \in \langle N \rangle} a_k\, e^{jk\frac{2\pi}{N}n}$$

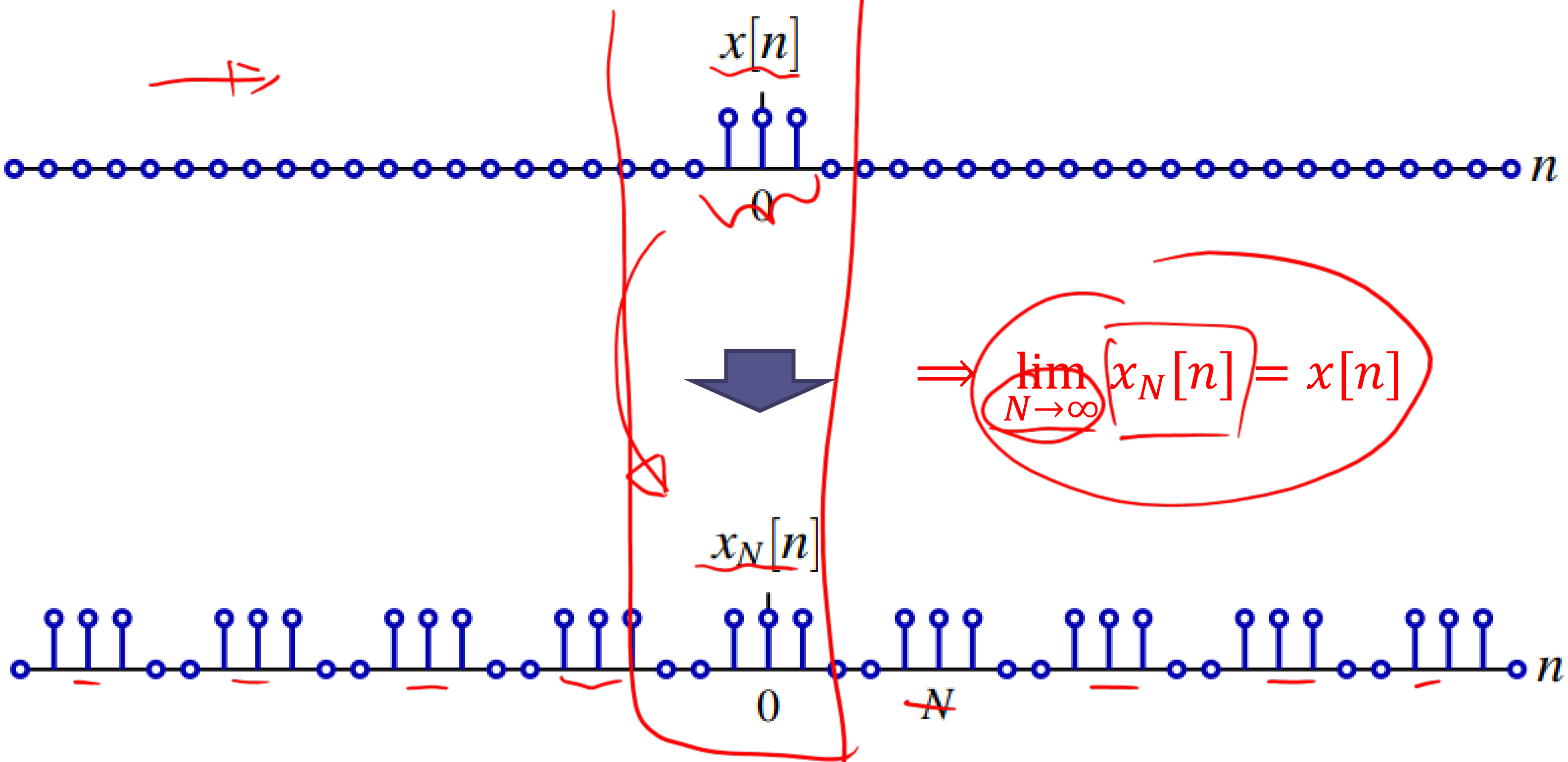
  - **Analysis equation**

$$a_k = \frac{1}{N} \sum_{n \in \langle N \rangle} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n \in \langle N \rangle} x[n] e^{-jk\frac{2\pi}{N}n}$$

DT Fourier Transform of Aperiodic Signal $x[n]$

General strategy

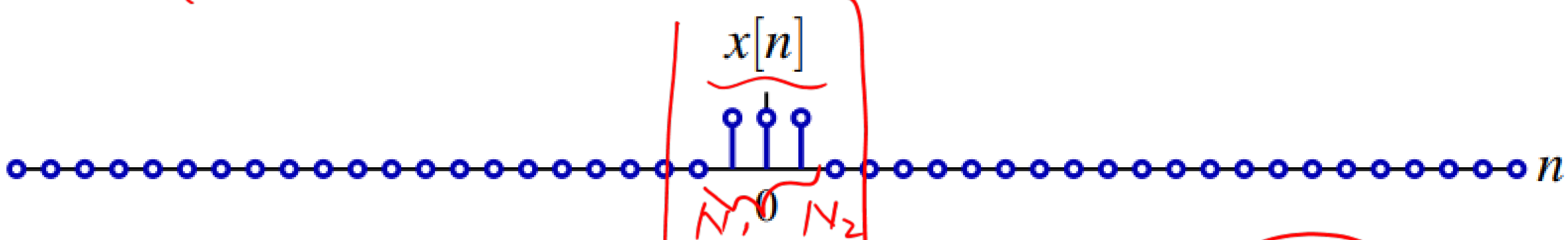
- approximate $x[n]$ by a periodic signal $x_N[n]$ with infinite period N



$$\Rightarrow \lim_{N\to\infty} x_N[n] = x[n]$$

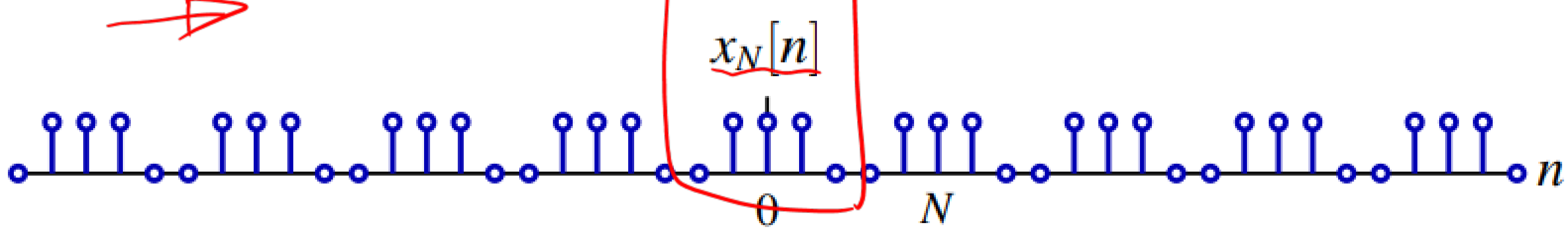
DT Fourier Transform

- **Step 1:**
  - Let $x[n]$ be a aperiodic DT signal with supp $x \subset [N_1, N_2]$



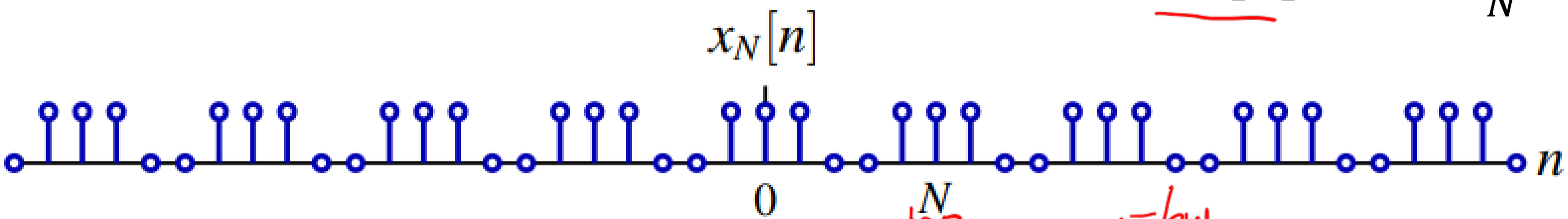
  - **Construct a periodic extension** $x_N[n]$ with period $N > \mathrm{N}_2 - \mathrm{N}_1 + 1$

$$x_N(n) = \sum_{k=-\infty}^{\infty} x[n + kN]$$



DT Fourier Transform

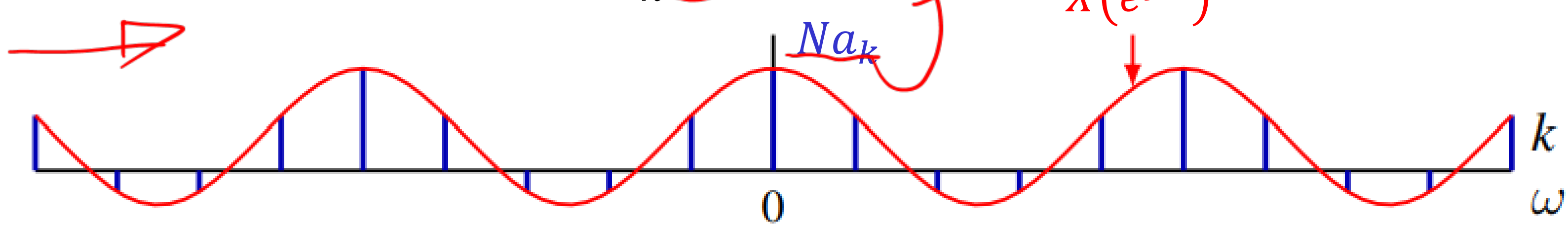
- **Step 2: Fourier series** representation for $x_N[n]$, $\omega_0 = \frac{2\pi}{N}$



$$a_k = \frac{1}{N}\sum_{n\in\langle N\rangle} x_N[n]e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=N_1}^{N_2} x[n]e^{-jk\omega_0 n} = \frac{1}{N}X(e^{jk\omega_0})$$

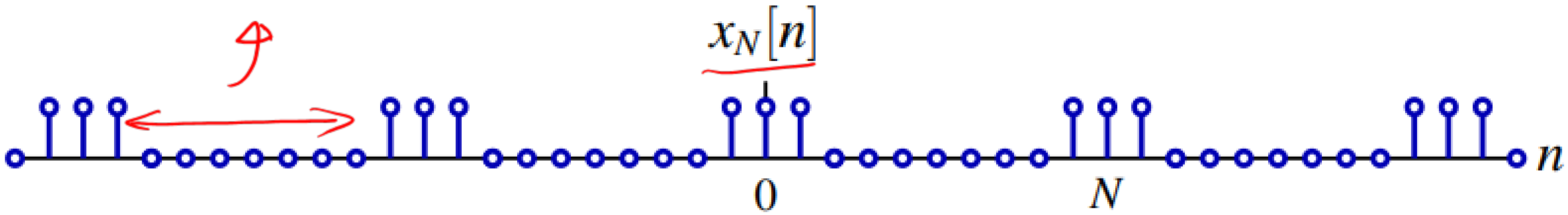
- where we define

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$



DT Fourier Transform

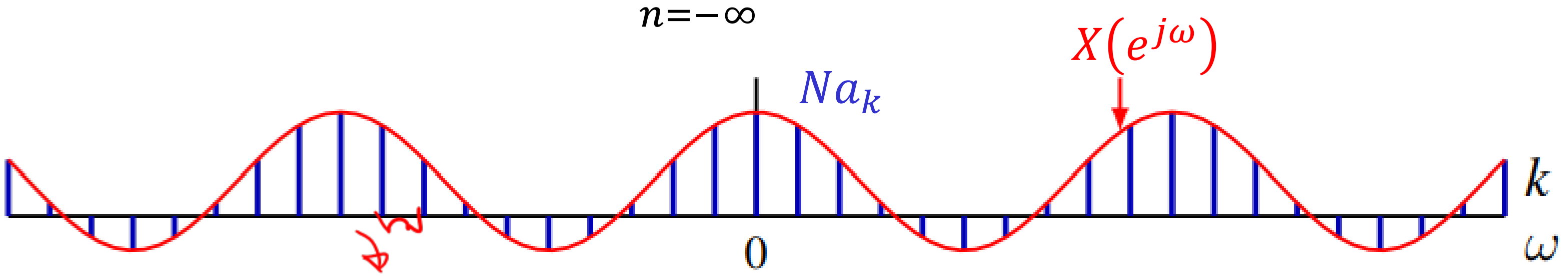
- As N increases, discrete frequencies are sampled more densely



$$a_k = \frac{1}{N}\sum_{n\in\langle N\rangle} x_N[n]e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=N_1}^{N_2} x[n]e^{-jk\omega_0 n} = \frac{1}{N}X(e^{jk\omega_0})$$

  - where we define

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$



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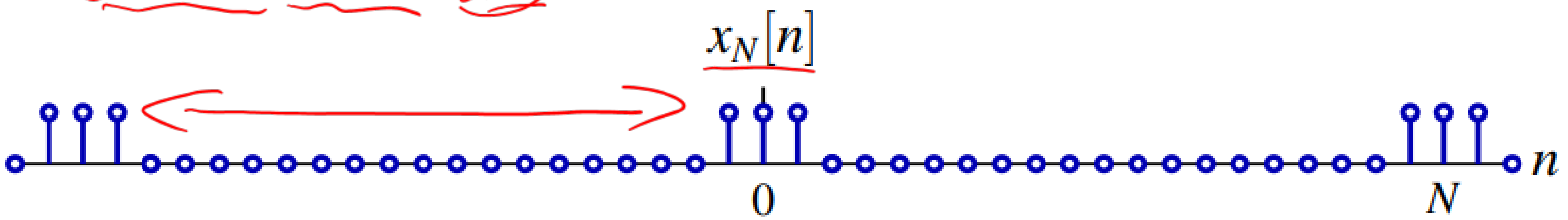
DT Fourier Transform

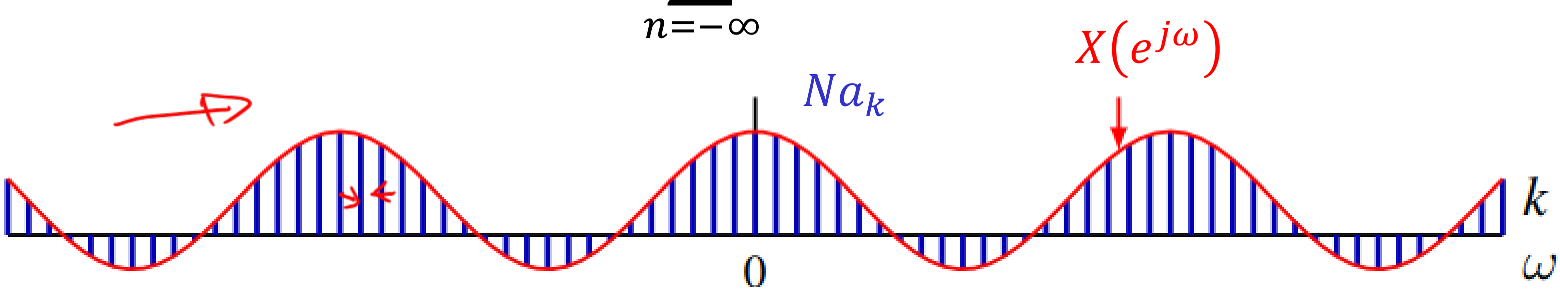
- As $N \to \infty$, $\omega_0 \to 0$, Na_k approaches **envelope $X(e^{j\omega})$**



$$a_k = \frac{1}{N}\sum_{n\in\langle N\rangle} x_N[n]e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=N_1}^{N_2} x[n]e^{-jk\omega_0 n} = \frac{1}{N}X(e^{jk\omega_0})$$

  - where we define

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$



DT Fourier Transform

- **Step 3: Synthesis equation** for DT Fourier series of $x_N[n]$,

$$x_N[n] = \sum_{k\in\langle N\rangle} a_k\, e^{jk\omega_0 n} = \sum_{k\in\langle N\rangle} \frac{1}{N} X(e^{jk\omega_0})\, e^{jk\omega_0 n}$$

since $\omega_0 = 2\pi/N$,

$$x_N[n] = \frac{1}{2\pi} \sum_{k\in\langle N\rangle} X(e^{jk\omega_0})\, e^{jk\omega_0 n}\omega_0$$

As $N \to \infty \implies \omega_0 \to 0, x_N[n] \to x[n], \omega_0 \to d\omega, \Sigma \to \int$

$$x[n] = \lim_{N\to\infty} x_N[n] = \lim_{\omega_0\to 0} \frac{1}{2\pi} \sum_{k\in\langle N\rangle} X(e^{jk\omega_0})\, e^{jk\omega_0 n}\omega_0$$

$$= \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Integration can take any period of length 2π

DT Fourier Transform Pair

- **DT Fourier transform (analysis equation)**

$$X(e^{j\omega}) = \mathcal{F}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

  - where $X(e^{j\omega})$ called **spectrum** of $x[n]$, **periodic** with period 2π

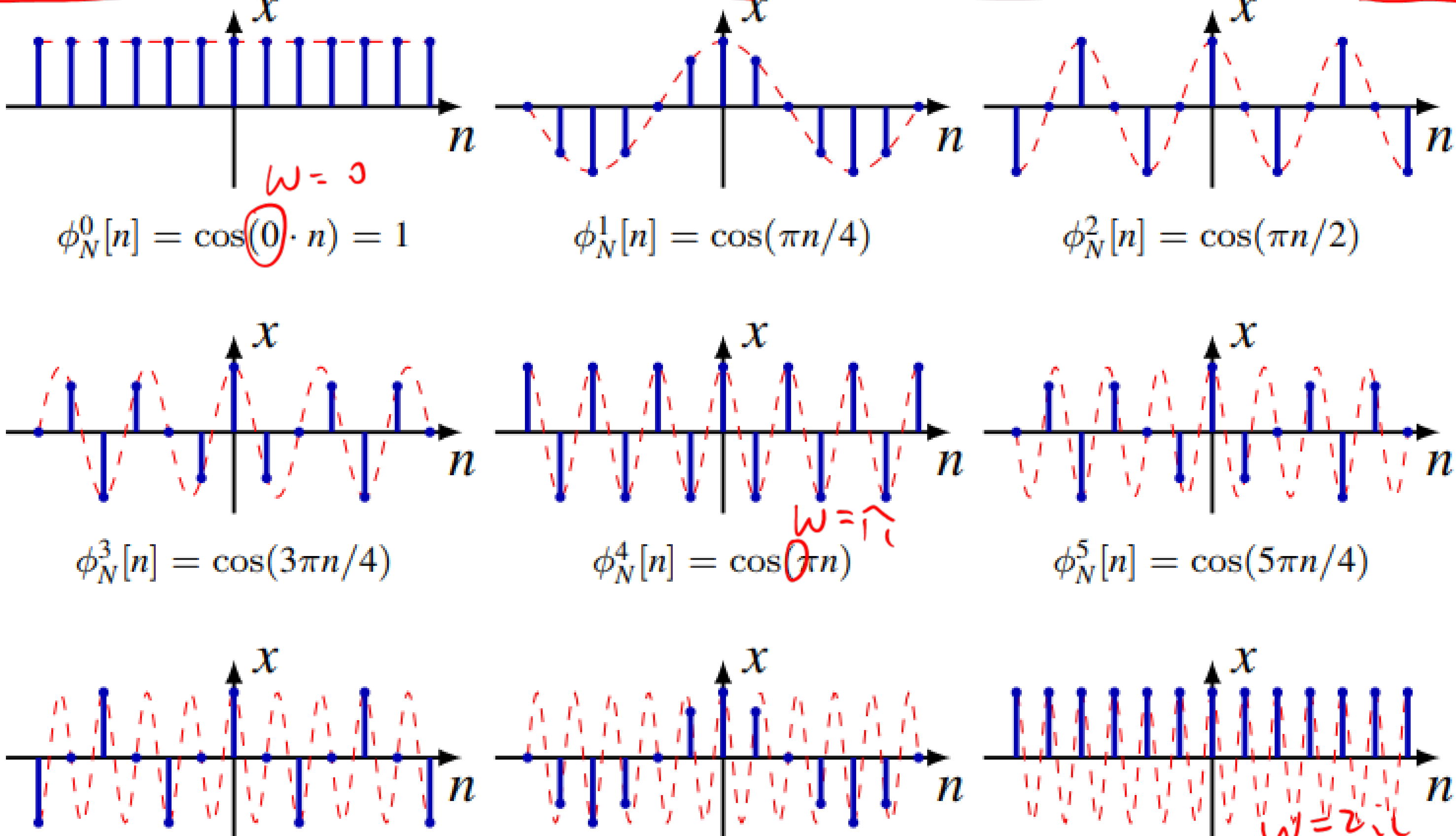
- **DT inverse Fourier transform (synthesis equation)**

$$x[n] = \mathcal{F}^{-1}\{X(e^{j\omega})\} = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

  - frequency component $e^{j\omega n}$ has "amplitude" of $X(e^{j\omega})d\omega/2\pi$
  - integrate over **a frequency interval** producing **distinct** $e^{j\omega n}$

High vs. Low Frequencies for DT Signals

- High frequencies around $(2k+1)\pi$, low frequencies around $2k\pi$



$\phi_N^0[n] = \cos(0 \cdot n) = 1$ $\quad$ $\phi_N^1[n] = \cos(\pi n/4)$ $\quad$ $\phi_N^2[n] = \cos(\pi n/2)$

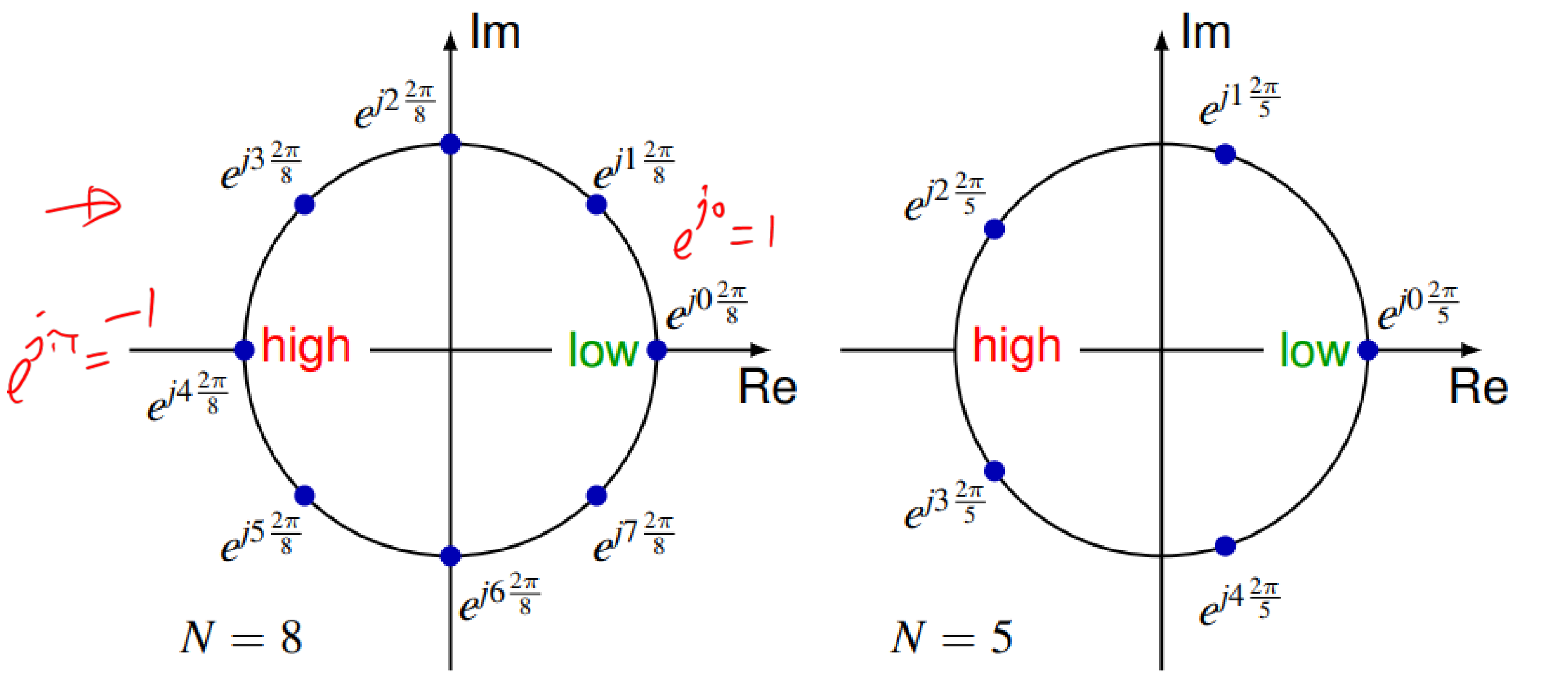
$\phi_N^3[n] = \cos(3\pi n/4)$ $\quad$ $\phi_N^4[n] = \cos(\pi n)$ $\quad$ $\phi_N^5[n] = \cos(5\pi n/4)$

$\phi_N^6[n] = \cos(3\pi n/2)$ $\quad$ $\phi_N^7[n] = \cos(7\pi n/4)$ $\quad$ $\phi_N^8[n] = \cos(2\pi n) = 1$

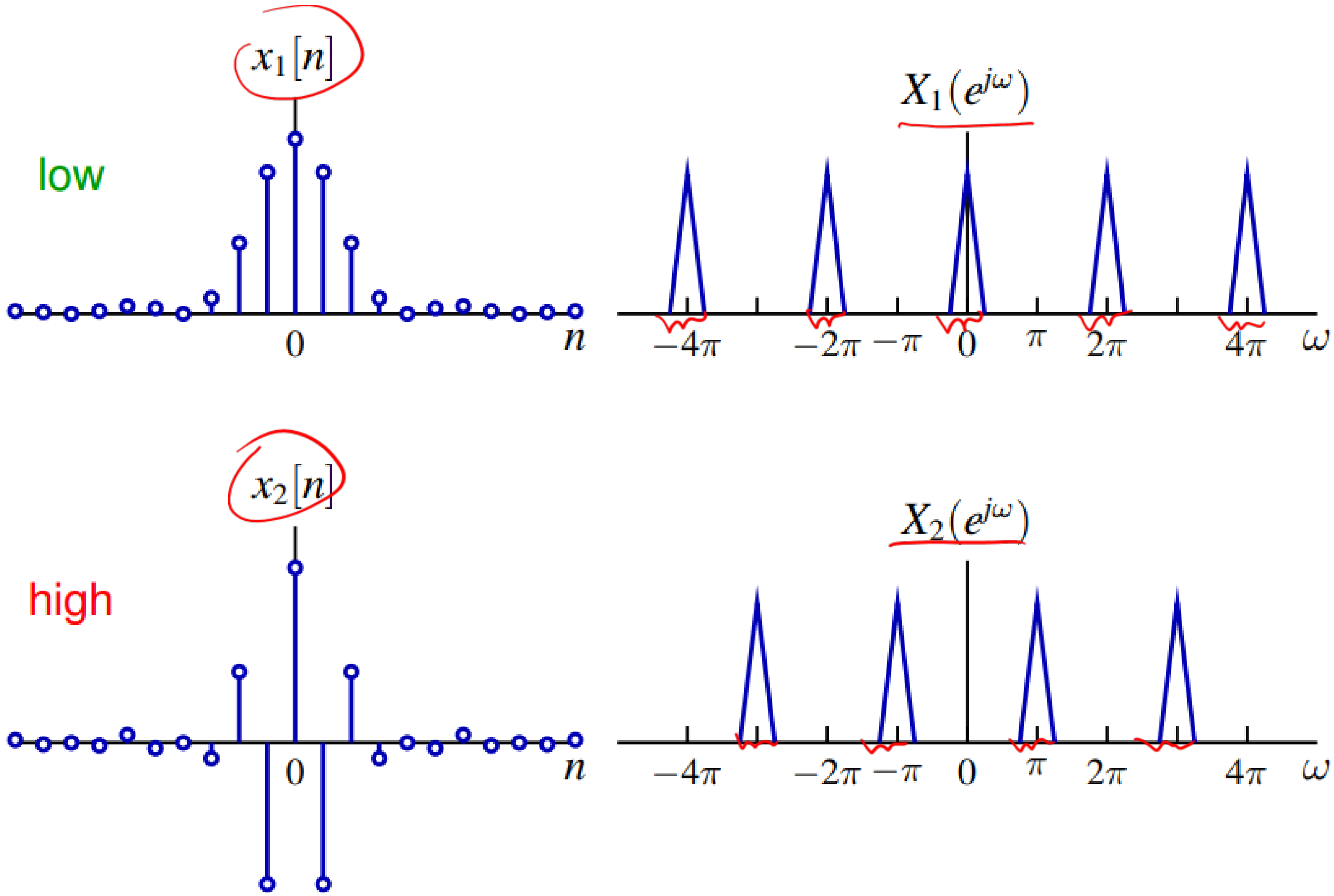
High vs. Low Frequencies for DT Signals

$e^{j\omega n}$, $n = 0, 1, \cdots, N-1$

- Discrete frequencies of periodic signals with period N
  - evenly spaced points on unit circle
  - low frequencies close to 1, high frequencies close to -1



High vs. Low Frequencies for DT Signals



CT Fourier Series of Periodic Signal $x_T(t)$

- **Fourier Transform** of a finite duration signal $x[n]$ that is equal to $x_N[n]$ over one period, e.g., for $N_1 \le n \le N_2$

$$X(e^{j\omega}) = \mathcal{F}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- **Fourier coefficients** of periodic signal $x_N[n]$ with period N

$$a_k = \frac{1}{N}\sum_{n\in\langle N\rangle} x_N[n]e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=N_1}^{N_2} x_N[n]e^{-jk\omega_0 n}$$

$$= \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n]e^{-jk\omega_0 n} = \frac{1}{N} X(e^{j\omega})\Big|_{\omega=k\omega_0}$$

$\Longrightarrow$ **proportional** to equally spaced **samples** of the **Fourier transform** of $x[n]$, i.e., one period of $x_N[n]$

Convergence of DT Fourier Transform

$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$ finite infinite

- **Analysis equation** will **converge** if
  - $x[n]$ is **absolutely summable**

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

  - or $x[n]$ has **finite energy**

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

2π

- **Synthesis equation** in general has **no convergence issues**, since the integration is taken over a finite interval

Example

- **Unit impulse**

$$x[n] = \delta[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega}) = 1$$

- **DC signal**

$$x[n] = 1 \xleftrightarrow{\mathcal{F}} X(e^{j\omega}) = 2\pi \sum_{l=-\infty}^{\infty} \delta[\omega - 2\pi l]$$

- **Proof:**

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{2\pi}\left(2\pi \sum_{l=-\infty}^{\infty} \delta[\omega - 2\pi l]\right) e^{j\omega n}d\omega$$

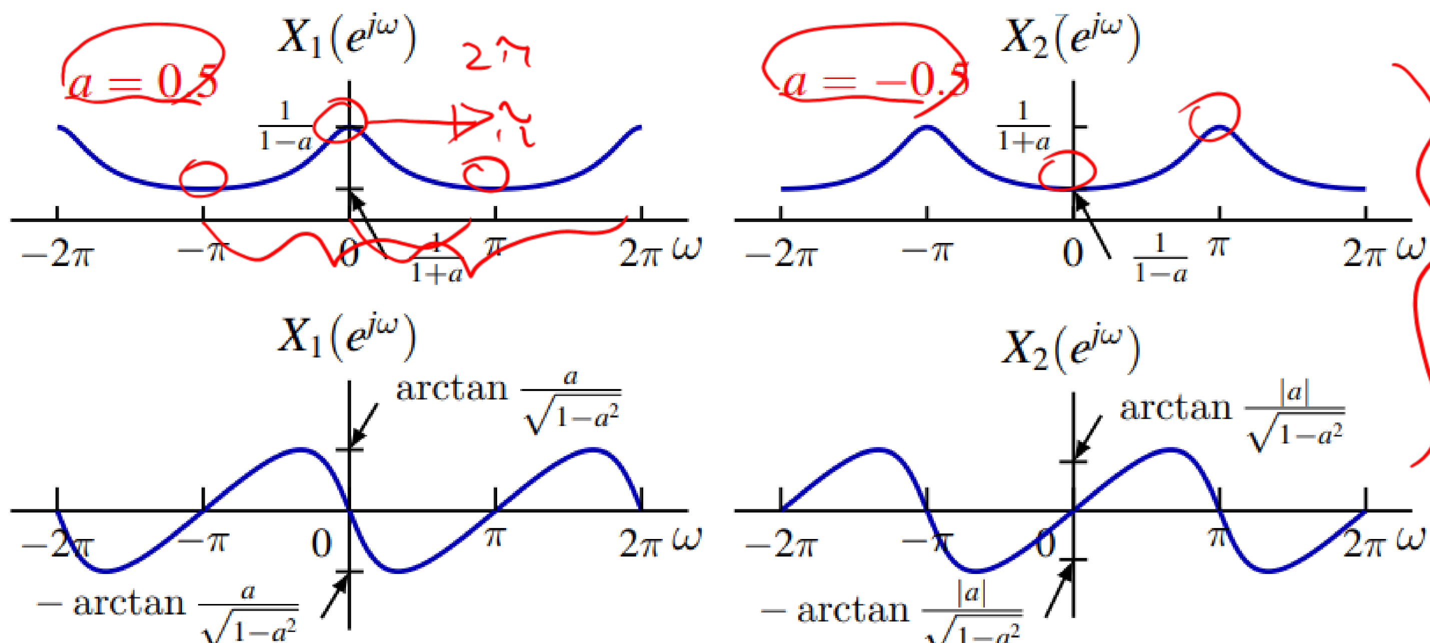
$$= \frac{1}{2\pi}\int^{\pi} 2\pi\delta(\omega)e^{j\omega n}d\omega = 1$$

Example

- **One-sided Decaying Exponential**

$$x[n] = a^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\omega}\right) = \frac{1}{1 - ae^{-j\omega}} \qquad |a| < 1$$

$$\left|X\left(e^{j\omega}\right)\right| = \frac{1}{\sqrt{1 - 2\,a\cos\omega + a^2}}, \qquad \arg X\left(e^{j\omega}\right) = -\arctan\frac{a\sin\omega}{1 - a\cos\omega}$$

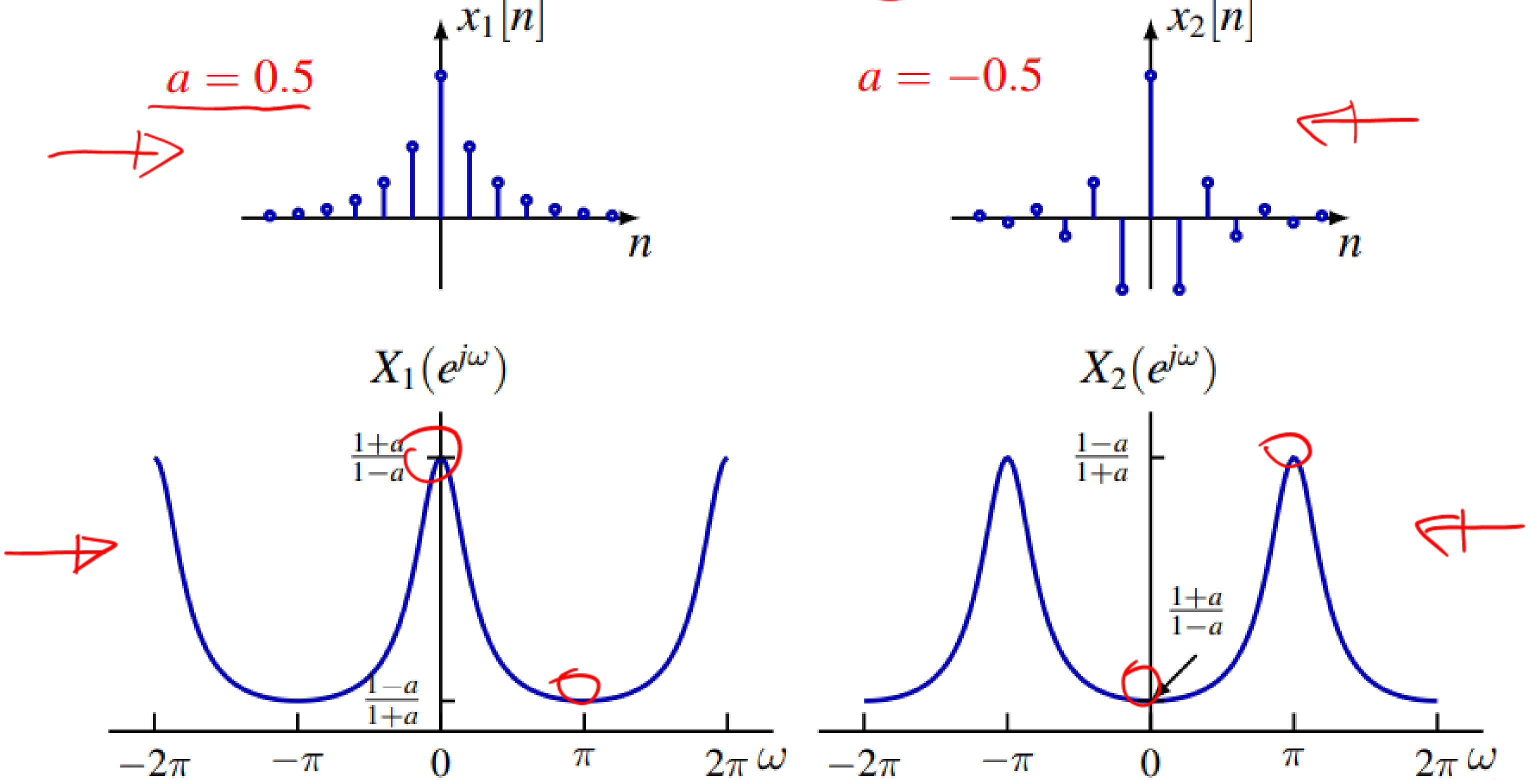


Example

$$\sum_{n=-\infty}^{\infty} a^{|n|} u[n] e^{-j\omega n} = \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} a^{n} e^{-j\omega n}$$

- **Two-sided Decaying Exponential**

$$x[n] = a^{|n|}u[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}) = \frac{1-|a|^2}{1-2a\cos\omega + a^2} \qquad |a| < 1$$



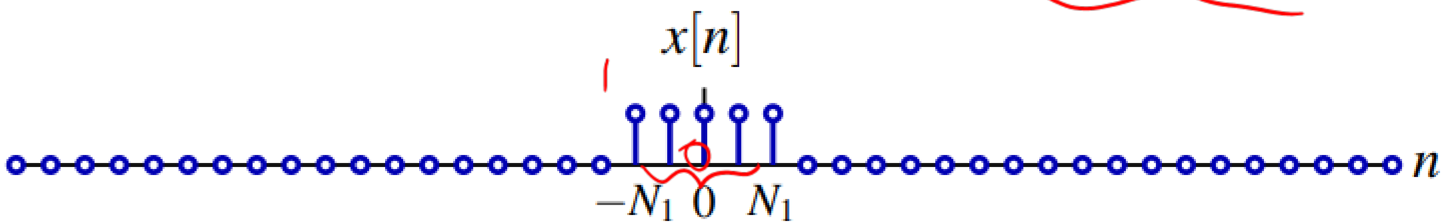
Example

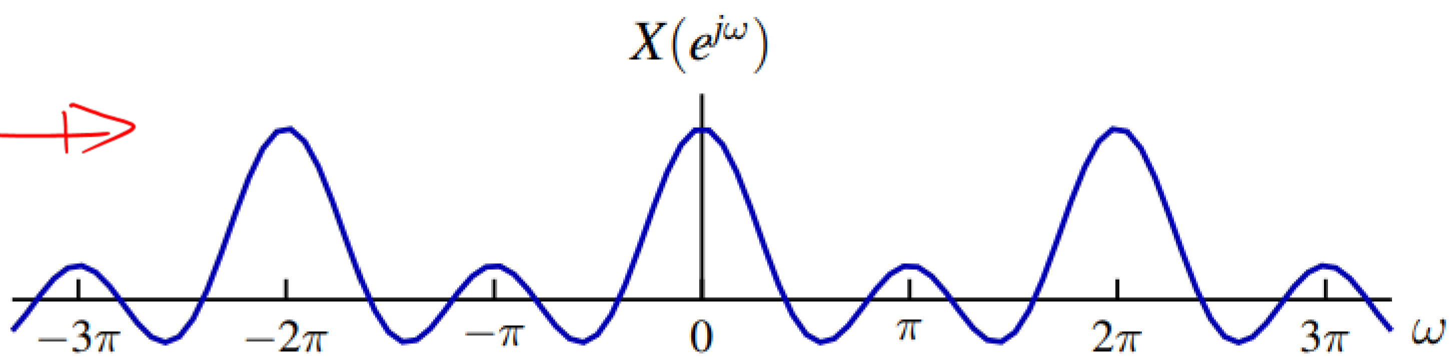
$\sum_{n=m}^{M} a^n = \frac{a^m - a^{M+1}}{1-a}$

$\sum_{n=-N_1}^{N_1} e^{-j\omega n} = \frac{e^{j\omega N_1} - e^{-j\omega N_1} e^{-j\omega}, \; e^{\frac{1}{2}j\omega}}{(1-e^{-j\omega})\, e^{\frac{1}{2}j\omega}}$

- **Rectangular Pulse**

$$x[n] = u[n + N_1] - u[n - N_1] \xleftrightarrow{\mathcal{F}} X(e^{j\omega}) = \frac{\sin\left(\frac{2N_1+1}{2}\omega\right)}{\sin\left(\frac{\omega}{2}\right)}$$





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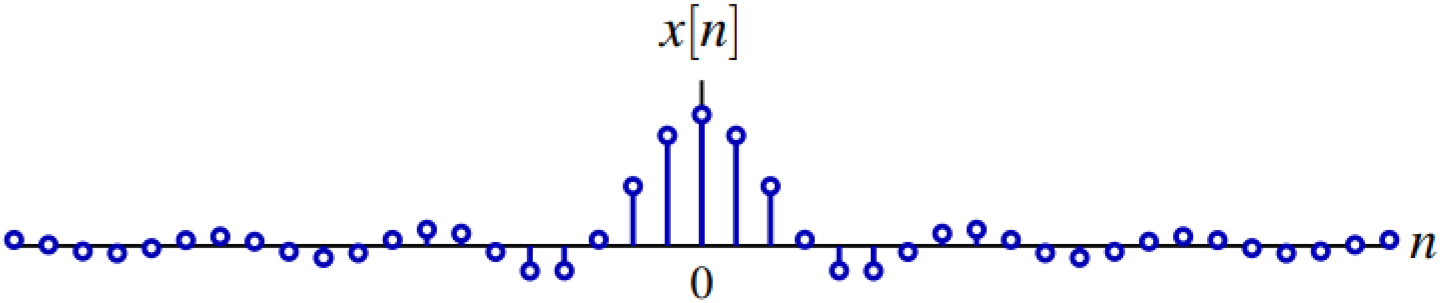
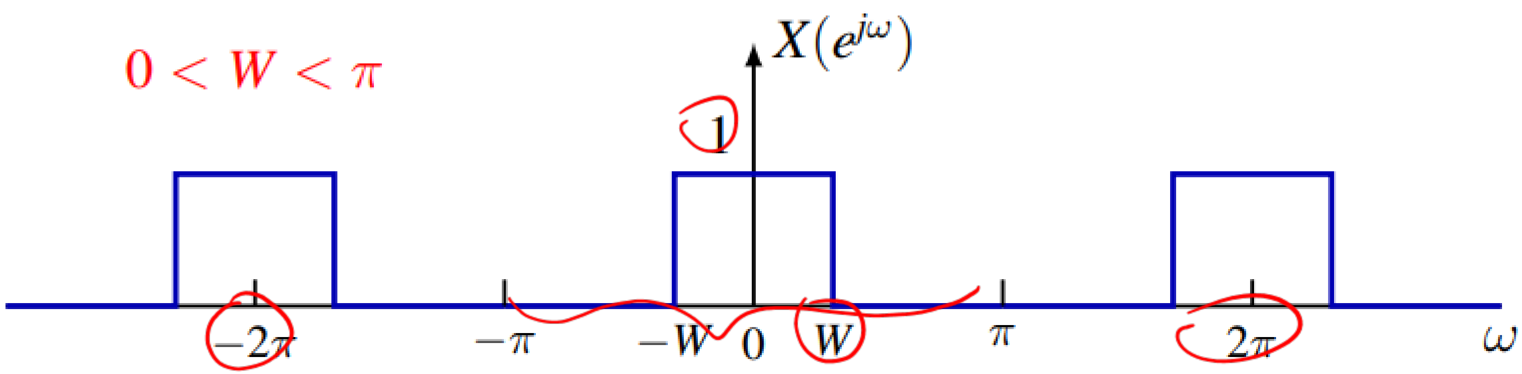
Example

- **Ideal Lowpass Filter**

$$x[n] = \frac{\sin(Wn)}{\pi n} \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}) = \begin{cases} 1, & 2k\pi - W \le \omega \le 2k\pi + W \\ 0, & 2k\pi + W \le \omega \le (2k+2)\pi - W \end{cases}$$



$0 < W < \pi$



DT Fourier Transform of Periodic Signals

- A periodic signal $x_N[n]$ with fundamental frequency $\omega_0 = 2\pi/N$ has a **DT Fourier series** representation

$$x_N[n] = \sum_{k\in\langle N\rangle} a_k\, e^{jk\omega_0 n}$$

$$\Longrightarrow X_N(e^{j\omega}) = \mathcal{F}\{x_N[n]\} = \sum_{k\in\langle N\rangle} a_k\, \mathcal{F}\{e^{jk\omega_0 n}\}$$

- Question then becomes

$$\mathcal{F}\{e^{jk\omega_0 n}\} = ?$$

DT Fourier Transform of Periodic Signals

- **Basic Fourier transform pair**

$$x[n] = e^{j\omega_0 n} \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}) = 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi l)$$

  - $X(e^{j\omega})$ has impulses at ω_0, $\omega_0 \pm 2\pi$, $\omega_0 \pm 4\pi$, ...

- Verified using the synthesis equation

$$\mathcal{F}^{-1}\{X(e^{j\omega})\} = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\delta(\omega - \omega_0)e^{j\omega n}d\omega = e^{j\omega_0 n} = x[n]$$

DT Fourier Transform of Periodic Signals

- **DT Fourier series** representation

$$x_N[n] = \sum_{k \in \langle N \rangle} a_k \, e^{jk\omega_0 n}$$

- with $e^{j\omega_0 n} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi l)$, then

$$X_N(e^{j\omega}) = \sum_{k=0}^{N-1} a_k \mathcal{F}\{e^{jk\omega_0 n}\} = \sum_{k=0}^{N-1} a_k \cdot 2\pi \left[\sum_{l=-\infty}^{\infty} \delta(\omega - k\omega_0 - 2\pi l) \right]$$

$$= 2\pi \sum_{l=-\infty}^{\infty} \sum_{k=0}^{N-1} a_k \delta\left(\omega - k\omega_0 - lN\omega_0\right)$$

$$= 2\pi \sum_{l=-\infty}^{\infty} \sum_{k=0}^{N-1} a_{k+lN} \delta\left(\omega - (k+lN)\omega_0\right) = 2\pi \sum_{m=-\infty}^{\infty} a_m \delta\left(\omega - m\omega_0\right)$$

DT Fourier Transform of Periodic Signals

$X(e^{j\omega}) = F\{x_N[n]\} = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$

- Can also verify using synthesis equation

$$\frac{1}{2\pi}\int_{-\frac{\pi}{N}}^{2\pi-\frac{\pi}{N}} X_N(e^{j\omega})e^{j\omega n}d\omega = \int_{-\frac{\pi}{N}}^{2\pi-\frac{\pi}{N}} \sum_{k=-\infty}^{\infty} a_k\delta(\omega - k\omega_0)e^{j\omega n}d\omega$$

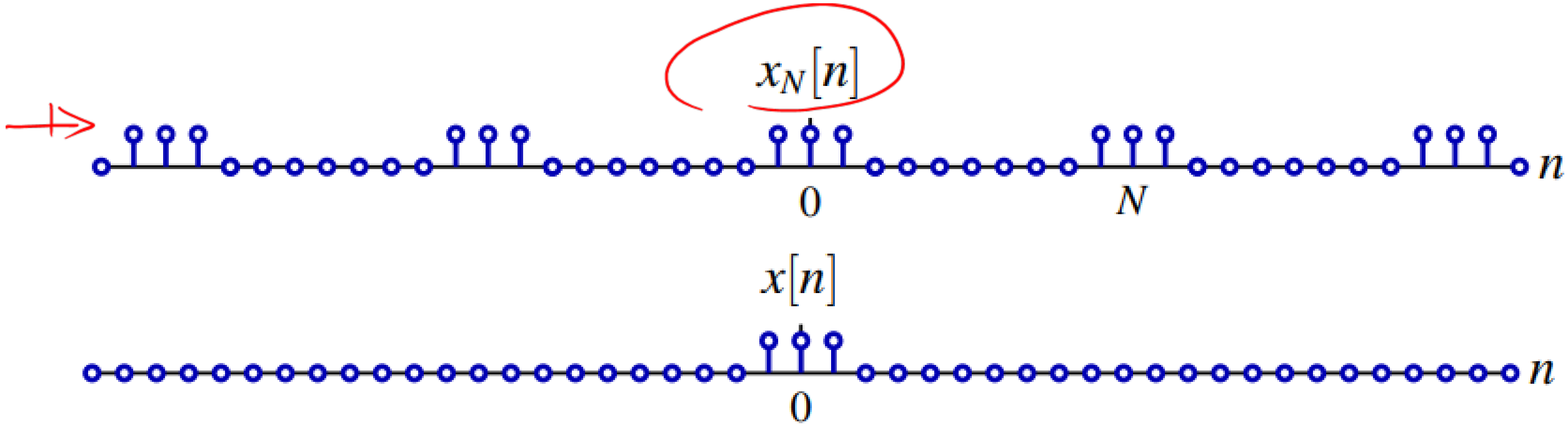
- Only terms with $k = 0, 1, \ldots, N-1$ in the interval of integration

$$\int_{-\frac{\pi}{N}}^{2\pi-\frac{\pi}{N}} \sum_{k=0}^{N-1} a_k\delta(\omega - k\omega_0)e^{j\omega n}d\omega = \sum_{k=0}^{N-1} a_k e^{jk\omega_0 n} = x_N[n]$$

- Therefore, verified

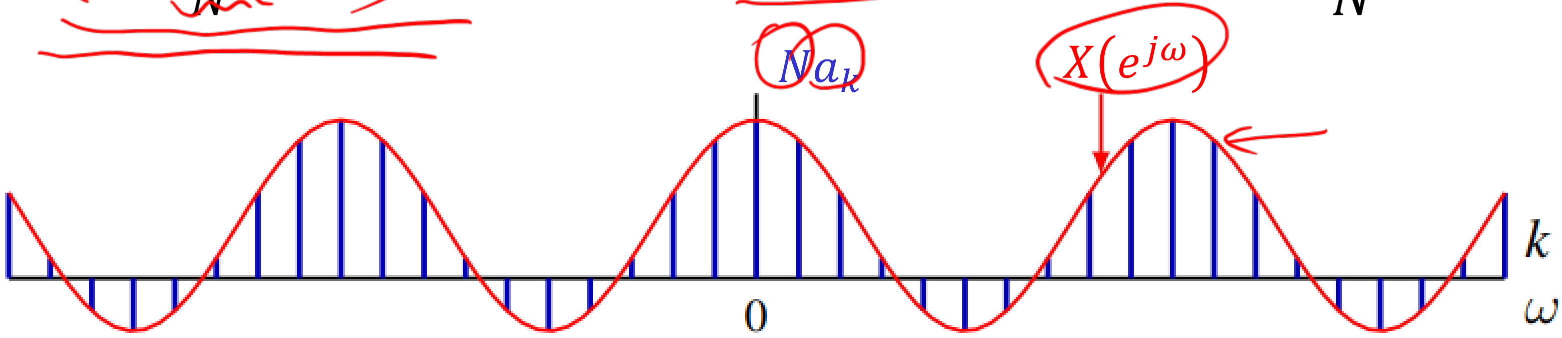
$$x_N[n] = \frac{1}{2\pi}\int_{2\pi} X_N(e^{j\omega})e^{j\omega n}d\omega$$

DT Fourier Transform of Periodic Signals

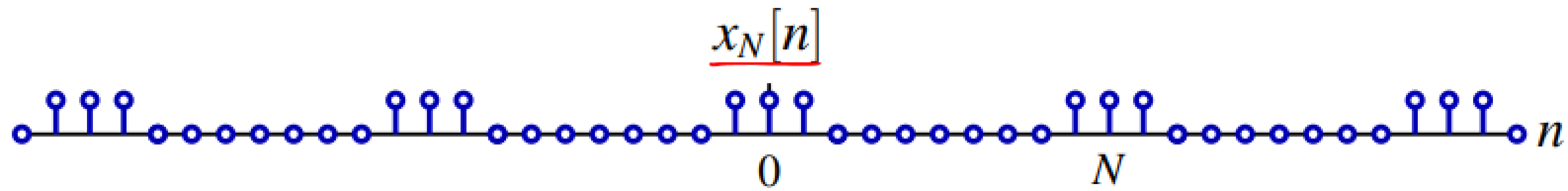


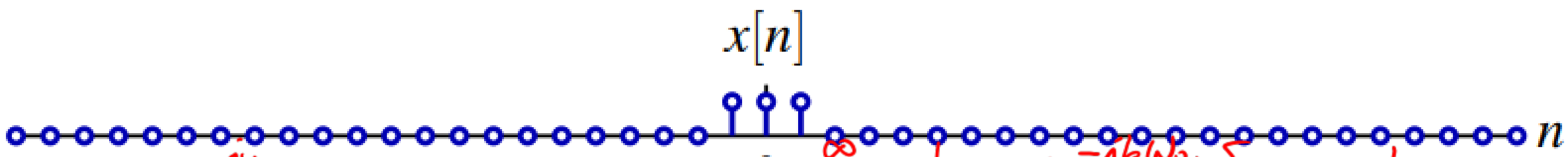
- Recall DTFS coefficient a_k is amplitude at frequency $\omega = k\omega_0$

$$a_k = \frac{1}{N} X\left(e^{jk\omega_0}\right), \qquad \text{where } X\left(e^{j\omega}\right) = \mathcal{F}\{x[n]\},\ \omega_0 = \frac{2\pi}{N}$$



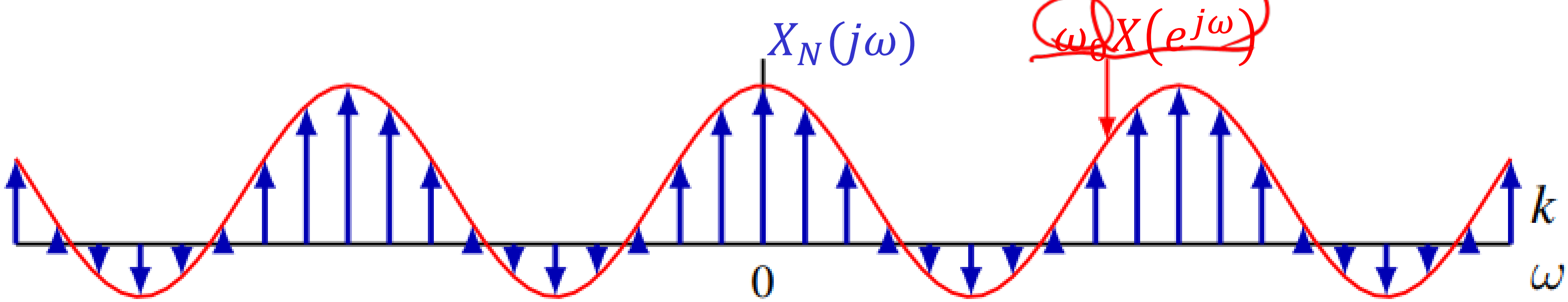
DT Fourier Transform of Periodic Signals





- DT Fourier transform $\frac{1}{2\pi}X_N\left(e^{jk\omega_0}\right)$ is density at frequency $\omega = k\omega_0$

$$X_N\left(e^{j\omega}\right) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\omega - k\omega_0\right) = \omega_0 \sum_{k=-\infty}^{\infty} X(jk\omega_0)\delta\left(\omega - k\omega_0\right)$$



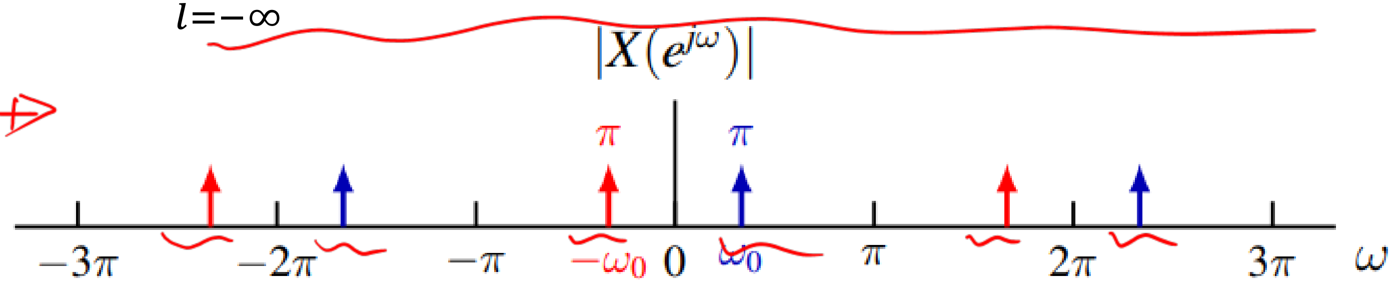
Example

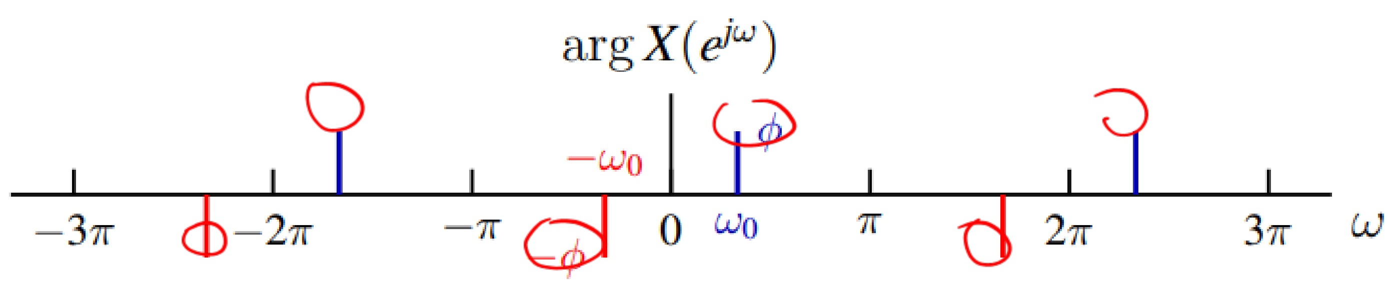
$$e^{j\omega_0 n} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\sum_{l=-\infty}^{\infty}\delta(\omega-\omega_0-2\pi l)$$

- **Sinusoids**

$$x[n] = \cos(\omega_0 n + \phi) = \frac{e^{j\phi}}{2}e^{j\omega_0 n} + \frac{e^{-j\phi}}{2}e^{-j\omega_0 n}$$

$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty}[\pi e^{j\phi}\delta(\omega-\omega_0-2l\pi) + \pi e^{-j\phi}\delta(\omega+\omega_0-2l\pi)]$$

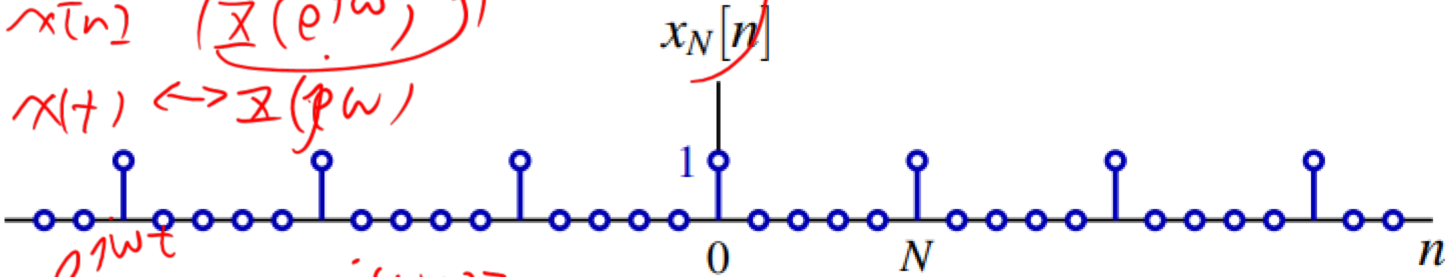


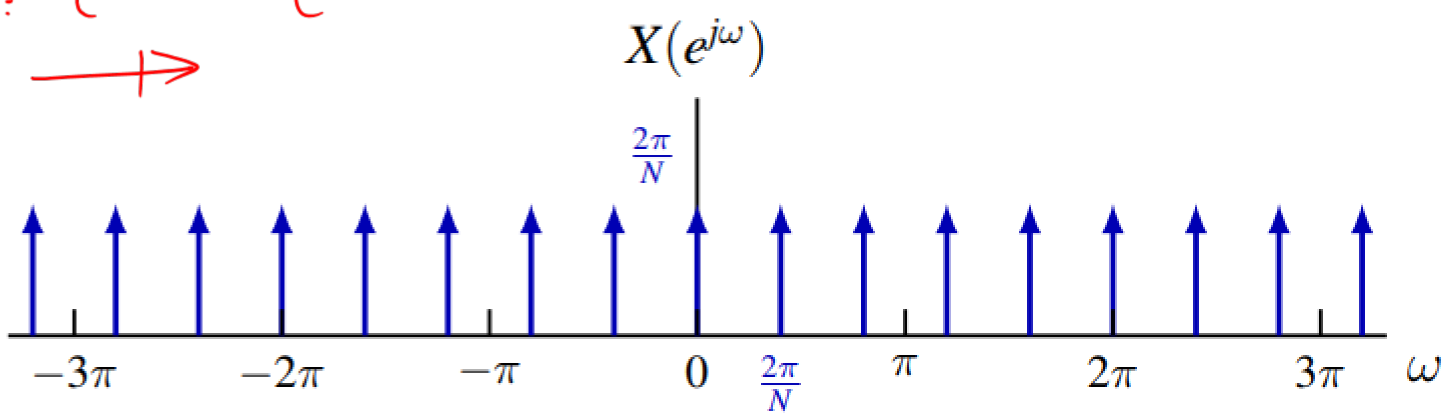


Example

- **DT Periodic Impulse Train**

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n-kN] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}) = \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta\left(\omega - k\frac{2\pi}{N}\right)$$





Properties of DT Fourier Transform

$$x[n]\overset{\mathcal{F}}{\longleftrightarrow}X\left(e^{j\omega}\right),\qquad y[n]\overset{\mathcal{F}}{\longleftrightarrow}Y\left(e^{j\omega}\right)$$

- **Periodicity (different from CTFT)**

$$X\left(e^{j(\omega+2\pi)}\right)=X\left(e^{j\omega}\right)$$

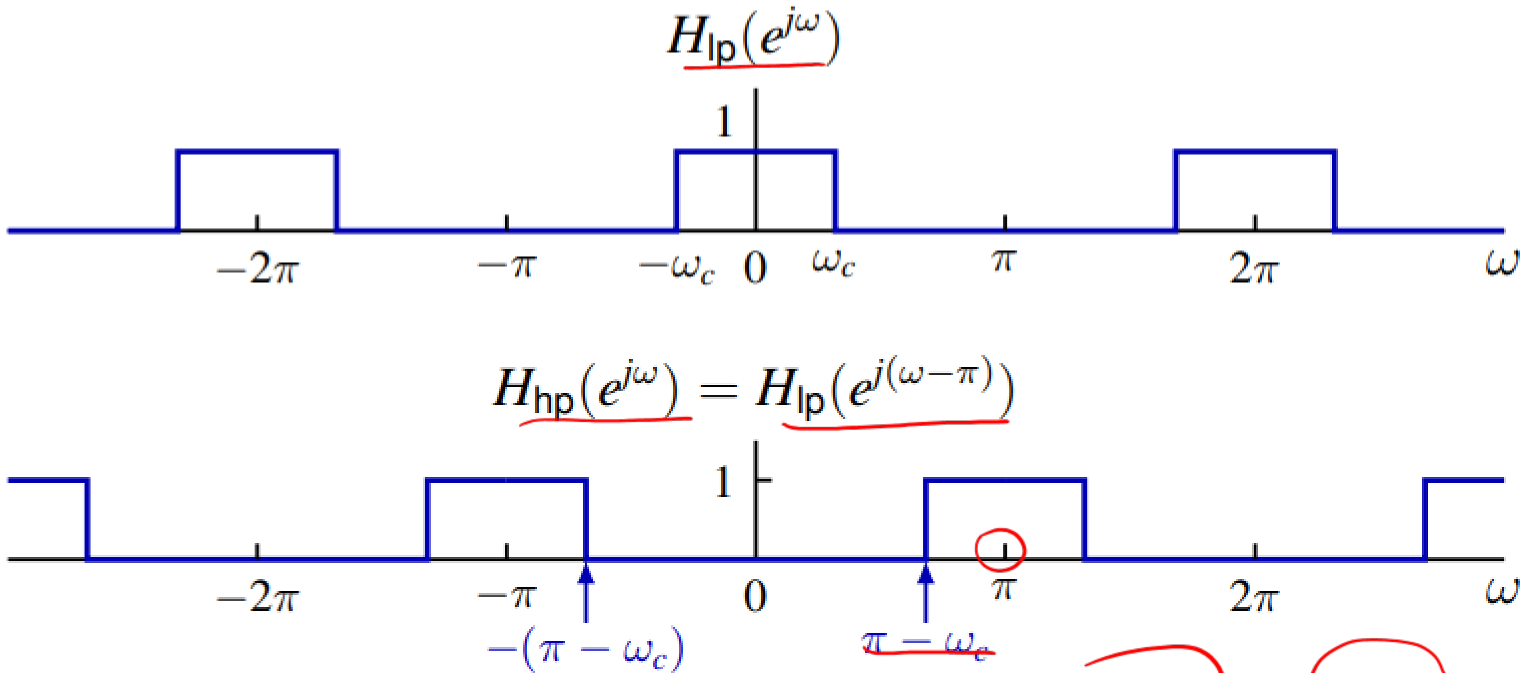
- **Linearity**

$$ax[n]+by[n]\overset{\mathcal{F}}{\longleftrightarrow}aX\left(e^{j\omega}\right)+bY\left(e^{j\omega}\right)$$

- **Time and frequency shifting**

$$x[n-n_0]\overset{\mathcal{F}}{\longleftrightarrow}e^{-j\omega n_0}X\left(e^{j\omega}\right),\qquad e^{j\omega_0 n}x[n]\overset{\mathcal{F}}{\longleftrightarrow}X\left(e^{j(\omega-\omega_0)}\right)$$

Example: Highpass vs. Lowpass Filters



$$H_{\text{hp}}\left(e^{j\omega}\right) = H_{\text{lp}}\left(e^{j(\omega-\pi)}\right) \Leftrightarrow h_{\text{hp}}[n] = e^{j\pi n} h_{\text{lp}}[n] = (-1)^n h_{\text{lp}}[n]$$

- Highpass filtering $y[n] = x[n] * h_{\text{hp}}[n]$ implemented by lowpass filter

(1) $x_1[n] = (-1)^n x[n]$, **(2)** $y_1[n] = x_1[n] * h_{\text{lp}}[n]$, **(3)** $y[n] = (-1)^n y_1[n]$

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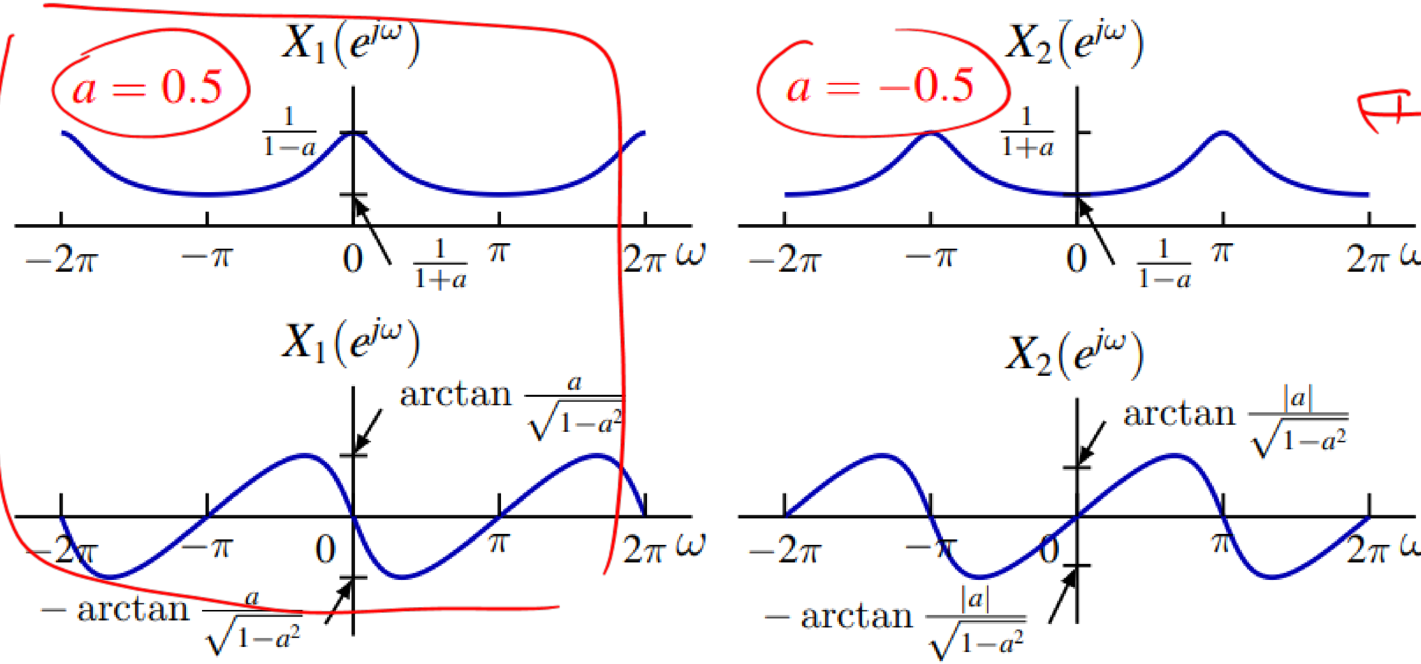
Example

- **One-sided Decaying Exponential**

$$x[n] = a^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} \quad |a| < 1$$

$$|X(e^{j\omega})| = \frac{1}{\sqrt{1 - 2a\cos\omega + a^2}}, \qquad \arg X(e^{j\omega}) = -\arctan\frac{a\sin\omega}{1 - a\cos\omega}$$

$x[n] = (-0.5)^n u[n] = (-1)^n (0.5)^n u[n]$, $e^{j\pi n}$



Properties of DT Fourier Transform

- **Time reversal**

$$x[-n] \stackrel{\mathcal{F}}{\longleftrightarrow} X\left(e^{-j\omega}\right)$$

- **Conjugation**

$$x^*[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X^*\left(e^{-j\omega}\right)$$

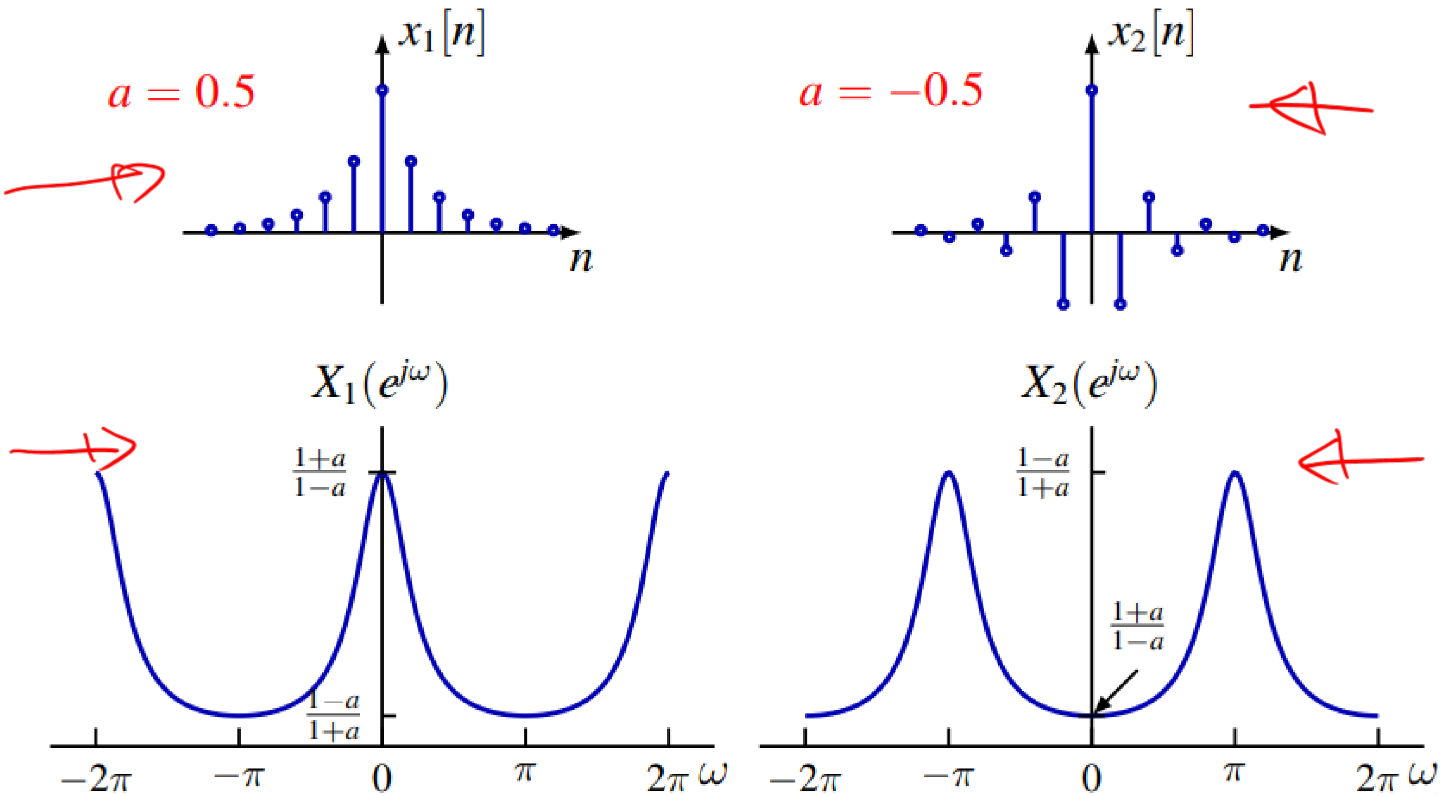
Proof: $\mathcal{F}\{x^*[n]\} = \sum_{n=-\infty}^{\infty} x^*[n]e^{-j\omega n} = \left[\sum_{n=-\infty}^{\infty} x[n]e^{j\omega n}\right]^* = X^*\left(e^{-j\omega}\right)$

- **Conjugate Symmetry**
  - $x[n]$ real $\Leftrightarrow X(e^{-j\omega}) = X^*(e^{j\omega})$
  - $x[n]$ even $\Leftrightarrow X(e^{j\omega})$ even, $x[n]$ odd $\Leftrightarrow X(e^{j\omega})$ odd
  - $x[n]$ real and even $\Leftrightarrow X(e^{j\omega})$ real and even
  - $x[n]$ real and odd $\Leftrightarrow X(e^{j\omega})$ purely imaginary and odd

Example

- **Two-sided Decaying Exponential**

$$x[n] = a^{|n|}u[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}) = \frac{1-|a|^2}{1-2\,a\cos\omega + a^2} \qquad |a| < 1$$



Differencing and Accumulation

- **First (backward) difference**

$$x[n]-x[n-1]\overset{\mathcal{F}}{\longleftrightarrow}(1-e^{-j\omega})X(e^{j\omega})$$

- **Accumulation (Running sum)**

$$\sum_{m=-\infty}^{n}x[m]\overset{\mathcal{F}}{\longleftrightarrow}\frac{1}{1-e^{-j\omega}}X(e^{j\omega})+\pi X(e^{j0})\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)$$

  - first term from differencing property
  - second term is DTFT of DC component $\mathcal{F}\{\bar{x}\}$, $\bar{x}=\frac{1}{2}X(e^{j0})$

- **Example:** since $\delta[n]\overset{\mathcal{F}}{\longleftrightarrow}1$,

$$u[n]=\sum_{m=-\infty}^{n}\delta[m]\overset{\mathcal{F}}{\longleftrightarrow}\frac{1}{1-e^{-j\omega}}+\pi\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)$$

Time Expansion

- Recall **time and frequency scaling** property of CTFT

$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right), a \neq 0$$

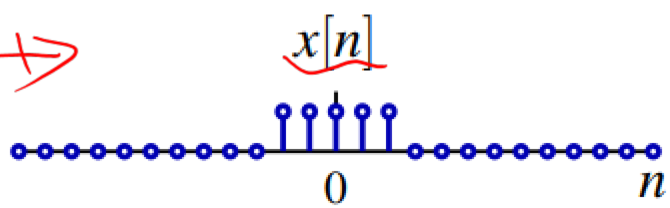
- But in DT case,
  - $x[an]$ makes no sense if $a \notin \mathbb{Z}$
  - for $a \in \mathbb{Z}$, if $a \neq \pm 1$, problems still exist,
    - e.g., let $a = 2$, $x[2n]$ misses odd values of $x[n]$

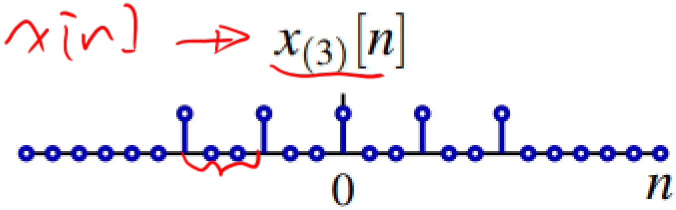
- We can "slow" a DT signal down by inserting consecutive zeros, for a positive integer k
  - $x_{(k)}[n]$ obtained by inserting $k - 1$ zeros between two successive values of $x[n]$

Time Expansion

- Formally, for a positive integer k, define $x_{(k)}[n]$ by

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is multiple of } k \\ 0, & \text{otherwise} \end{cases}$$





- If

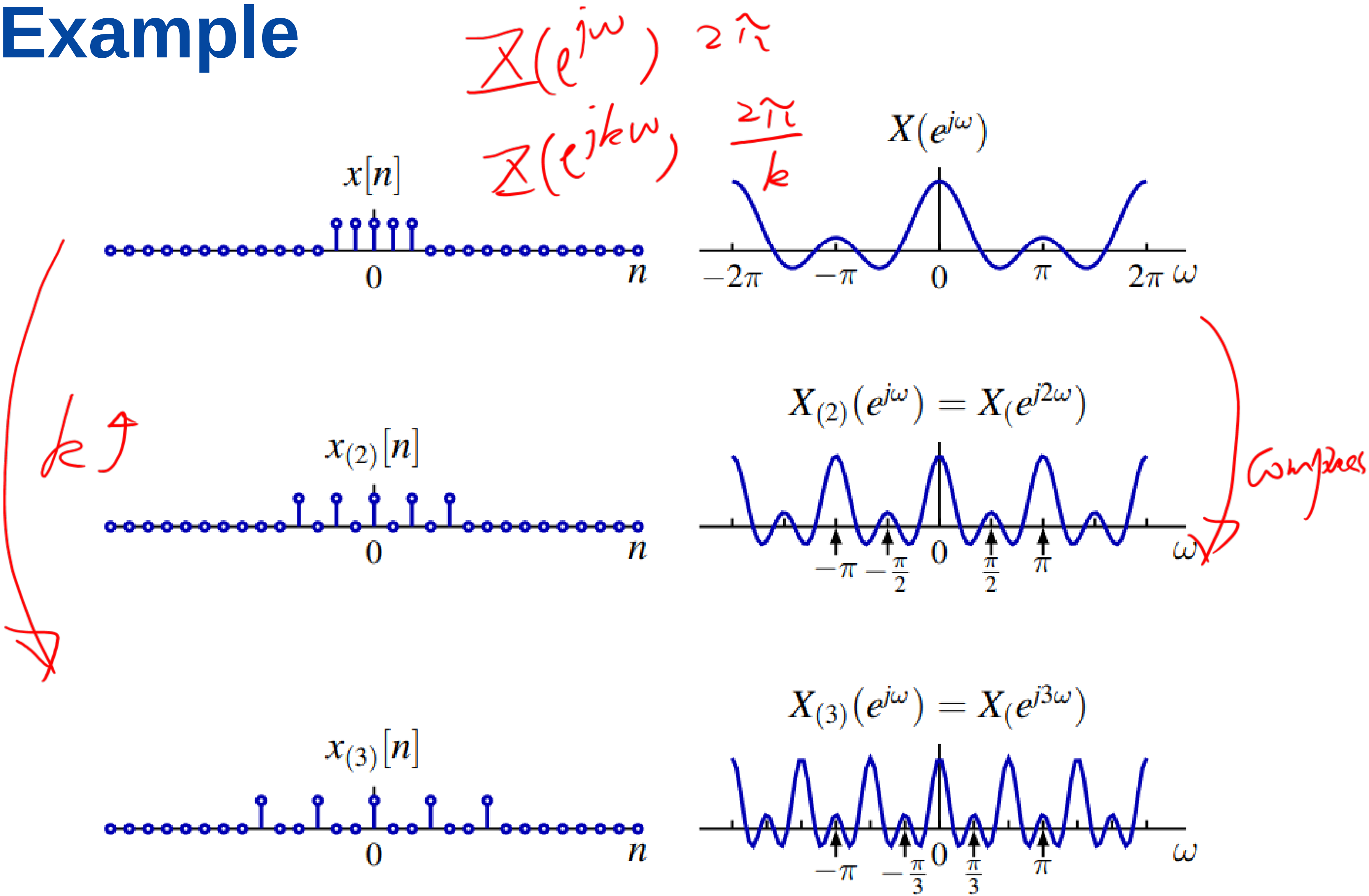
$$x[n] \overset{\mathcal{F}}{\longleftrightarrow}$$

then

$$x_{(k)}[n] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{jk\omega}\right)$$

- **Proof:** $\mathcal{F}\{x_{(k)}[n]\} = \sum_{n=-\infty}^{\infty} x_{(k)}[n] e^{-j\omega n} = \sum_{l=-\infty}^{\infty} x[l] e^{-j\omega kl} = X(e^{jk\omega})$

Example


$x[n]$
0
n
$X(e^{j\omega})$
-2π
$-\pi$
0
π
2π
ω
$x_{(2)}[n]$
0
n
$X_{(2)}(e^{j\omega}) = X(e^{j2\omega})$
$-\pi$
$-\frac{\pi}{2}$
0
$\frac{\pi}{2}$
π
ω
$x_{(3)}[n]$
0
n
$X_{(3)}(e^{j\omega}) = X(e^{j3\omega})$
$-\pi$
$-\frac{\pi}{3}$
0
$\frac{\pi}{3}$
π
ω

Differentiation in Frequency

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$$

- **Differentiation in frequency**

$$nx[n] \overset{\mathcal{F}}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}$$

- **Proof:**

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

differentiate both sides,

$$\frac{d}{d\omega}X(e^{j\omega}) = -j\sum_{n=-\infty}^{\infty} nx[n]e^{-j\omega n}$$

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Parseval's Relation

For a DT Fourier transform pair $x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} \left|X(e^{j\omega})\right|^2 d\omega = \int_{2\pi} \left|X(e^{j\omega})\right|^2 \frac{d\omega}{2\pi}$$

- **Note:** ω is angular frequency and $f = \omega/2\pi$ is frequency
- **Interpretation: Energy conservation**
  - $\sum_{n=-\infty}^{\infty} |x[n]|^2$: total energy
  - $\left|X(e^{j\omega})\right|^2$: energy per unit frequency, called **energy-density spectrum**

Parseval's Relation

- For a DT Fourier transform pair $x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} \left|X(e^{j\omega})\right|^2 d\omega = \int_{2\pi} \left|X(e^{j\omega})\right|^2 \frac{d\omega}{2\pi}$$

- **Proof:**

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} x[n]x^*[n] = \sum_{n=-\infty}^{\infty} x[n]\left[\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega n} d\omega\right]^*$$

$$= \sum_{n=-\infty}^{\infty} x[n]\left[\frac{1}{2\pi}\int_{2\pi} X^*(e^{j\omega})\, e^{-j\omega n} d\omega\right]$$

$$= \frac{1}{2\pi}\int_{2\pi} X^*(e^{j\omega})\left[\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right] d\omega$$

$$= \frac{1}{2\pi}\int_{2\pi} X^*(e^{j\omega})X(e^{j\omega})\, d\omega = \frac{1}{2\pi}\int_{2\pi} \left|X(e^{j\omega})\right|^2 d\omega$$

Convolution Property of DTFT

$$y[n] = x[n] * h[n] \overset{\mathcal{F}}{\longleftrightarrow} Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

- **Note:** dual of multiplication property of CTFS

$$x(t)y(t) \overset{\mathcal{FS}}{\longleftrightarrow} \sum_{m=-\infty}^{\infty} a_m b_{k-m}$$

- **Note:** Similar to CTFT, applicable when formula is well-defined

- **Proof:**

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[n] * h[n]e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x[k]h[n-k] \right) e^{-j\omega n}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \left(\sum_{n=-\infty}^{\infty} h[n-k]\, e^{-j\omega n} \right)$$

Frequency Response of LTI Systems

- Fully characterized by **impulse response** $h[n]$

$$y[n] = x[n] * h[n]$$

- Also fully characterized by **frequency response** $H(e^{j\omega}) = \mathcal{F}\{h[n]\}$, if $H(e^{j\omega})$ is well defined
  - BIBO stable system: $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$
  - other systems: e.g., accumulator $h[n] = u[n]$
- **Convolution property** implies
  - instead of computing $x[n] * h[n]$ in **time domain**, we can analyze a system in **frequency domain**

$$y[n] = x[n] * h[n]$$

$$\Downarrow$$

$$Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega}), \qquad H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = |H(e^{j\omega})| e^{j\angle H(e^{j\omega})}$$

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Example

- Determine the Response of an LTI system with impulse response $h[n] = a^n u[n]$, to the input $x[n] = b^n u[n]$, where $|a| < 1, |b| < 1$
- **Method 1: convolution** in time domain $y[n] = x[n] * h[n]$
- **Method 2:** solving the **difference equation with initial rest condition** $y[n] - ay[n-1] = b^n u[n]$
- **Method 3: Fourier transform**

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}, X(e^{j\omega}) = \frac{1}{1 - be^{-j\omega}} \Rightarrow Y(e^{j\omega}) = \frac{1}{(1 - ae^{-j\omega})(1 - be^{-j\omega})}$$

- If $a \neq b$, $Y(e^{j\omega}) = \frac{a/(a-b)}{1-ae^{-j\omega}} - \frac{b/(a-b)}{1-be^{-j\omega}} \Rightarrow y[n] = \frac{1}{a-b}(a^{n+1} - b^{n+1})u[n]$
- If $a = b$, $Y(e^{j\omega}) = \frac{j}{a} e^{j\omega} \frac{d}{d\omega}\left(\frac{1}{1-ae^{-j\omega}}\right) \Rightarrow y[n] = (n+1)a^n u[n]$

Example

- Determine the Response of an LTI system with impulse response $h[n] = a^n u[n]$, to the input $x[n] = \cos(\omega_0 n)$, where $|a| < 1$

real

- **Frequency response:**

$$H\left(e^{j\omega}\right) = \frac{1}{1 - ae^{-j\omega}}$$

- **Method 1:** using **eigenfunction property**

$H(e^{-j\omega_0}) = H^*(e^{j\omega_0})$

$$x[n] = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n}$$

$$y[n] = \frac{1}{2}H\left(e^{j\omega_0}\right)e^{j\omega_0 n} + \frac{1}{2}H\left(e^{-j\omega_0}\right)e^{-j\omega_0 n} = \mathcal{R}e\left\{H\left(e^{j\omega_0}\right)e^{j\omega_0 n}\right\}$$

$$= \frac{1}{\sqrt{1 - 2a\cos\omega_0 + a^2}}\cos\left(\omega_0 n - \arctan\frac{a\sin\omega_0}{1 - a\cos\omega_0}\right)$$

Example

$$1 \overset{\mathcal{F}}{\longleftrightarrow} 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

- **Method 2:** using **Fourier transform**

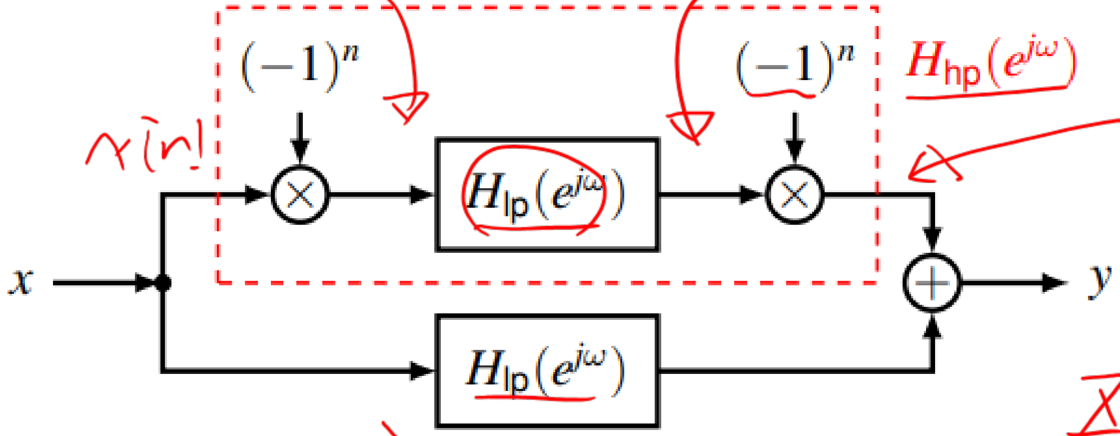
$$x[n] = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n}$$

$$\Longrightarrow X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} [\pi\delta(\omega - \omega_0 + 2k\pi) + \pi\delta(\omega + \omega_0 + 2k\pi)]$$

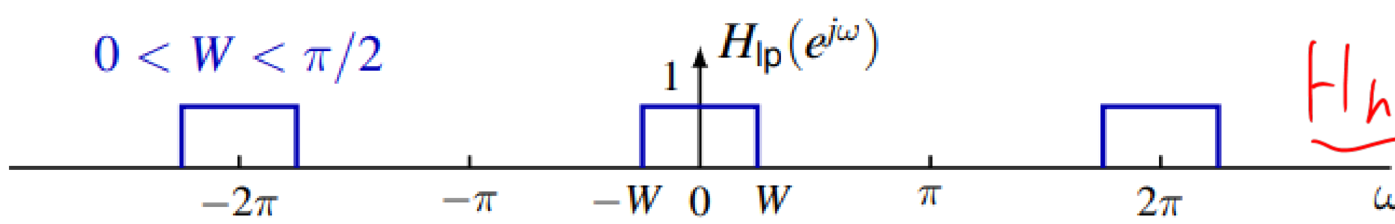
$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \pi H\left(e^{j(\omega_0 - 2k\pi)}\right)\delta(\omega - \omega_0 + 2k\pi) + \sum_{k=-\infty}^{\infty} \pi H\left(e^{-j(\omega_0 + 2k\pi)}\right)\delta(\omega + \omega_0 + 2k\pi)$$

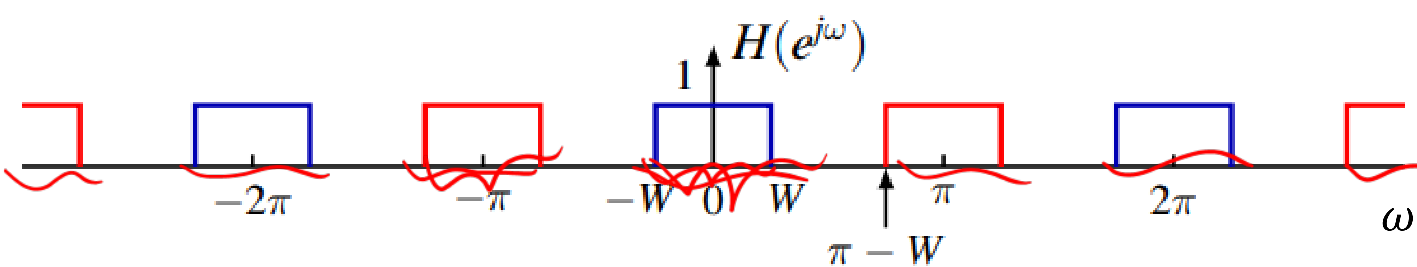
$$\Longrightarrow y[n] = \frac{1}{2}H(e^{j\omega_0})e^{j\omega_0 n} + \frac{1}{2}H(e^{-j\omega_0})e^{-j\omega_0 n}$$

Example: Ideal Bandstop Filter



$0 < W < \pi/2$





Multiplication Property of DTFT

$$x[n]\cdot y[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}\left[X(e^{j\omega}) \circledast Y(e^{j\omega})\right] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$$

- **Note:** dual of periodic convolution property of CTFS

$$x(t) \circledast y(t) \overset{\mathcal{FS}}{\longleftrightarrow} Ta_k b_k$$

- **Proof:**

$$\mathcal{F}\{x[n]y[n]\} = \sum_{n=-\infty}^{\infty} x[n]y[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})e^{j\theta n}d\theta\right)y[n]e^{-j\omega n}$$

$$= \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})\left(\sum_{n=-\infty}^{\infty} y[n]e^{-j(\omega-\theta)n}\right)d\theta$$

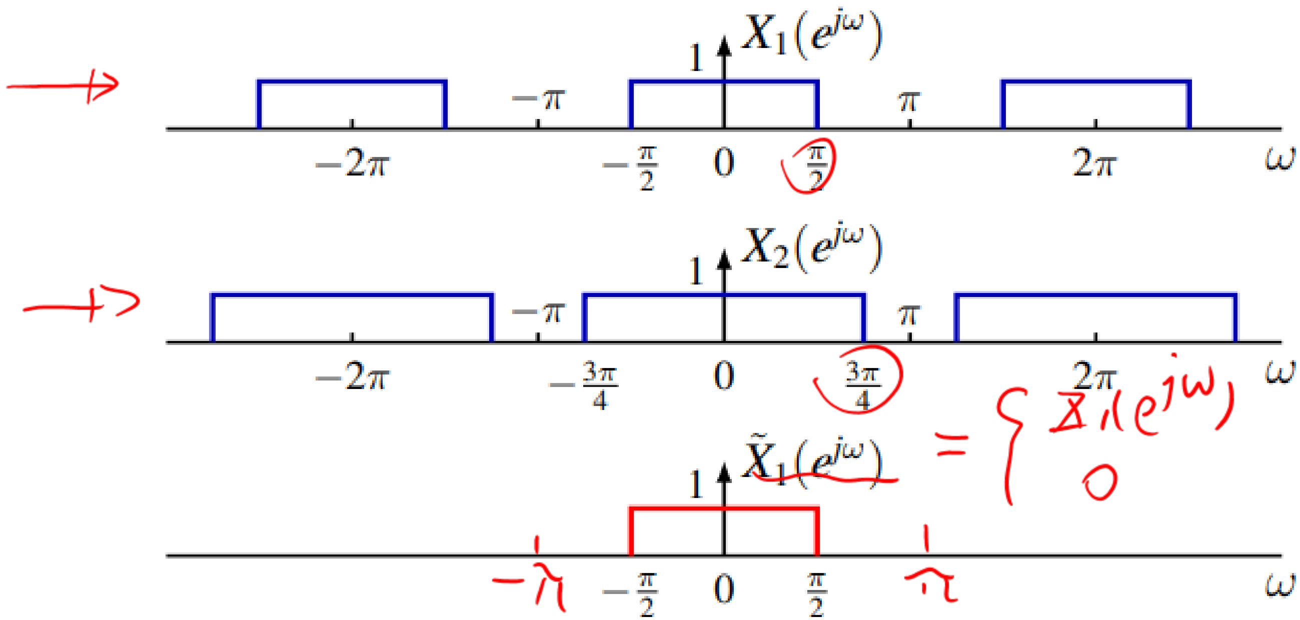
$$= \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$$

Example

- $x[n] = x_1[n]x_2[n]$, where $x_1[n] = \frac{\sin(3\pi n/4)}{\pi n}$, $x_2[n] = \frac{\sin(\pi n/2)}{\pi n}$

$$X(e^{j\omega}) = \frac{1}{2\pi}[X_1(e^{j\omega}) \circledast X_2(e^{j\omega})] = \frac{1}{2\pi}[\tilde{X}_1(e^{j\omega}) * X_2(e^{j\omega})]$$

periodic — aperiodic



$\tilde{X}_1(e^{j\omega}) = \begin{cases} X_1(e^{j\omega}) & \omega \in (-\pi, \pi) \\ 0 & \text{otherwise} \end{cases}$

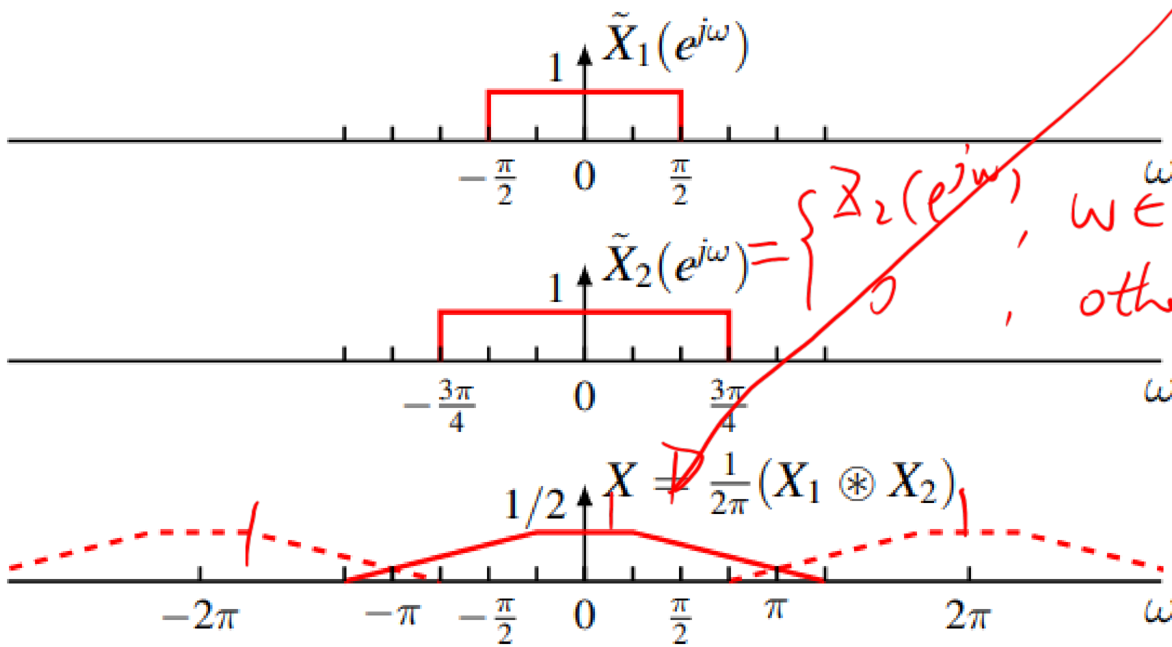
Example

- $x[n] = x_1[n]x_2[n]$, where $x_1[n] = \dfrac{\sin(3\pi n/4)}{\pi n}$, $x_2[n] = \dfrac{\sin(\pi n/2)}{\pi n}$

$$\frac{1}{2\pi}\left[X_1(e^{j\omega}) \circledast X_2(e^{j\omega})\right] = \frac{1}{2\pi}\left[\tilde{X}_1(e^{j\omega}) * X_2(e^{j\omega})\right] = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \tau_{2k\pi}\left[\tilde{X}_1(e^{j\omega}) * \tilde{X}_2(e^{j\omega})\right]$$

periodic **aperiodic** **aperiodic**

$\sum_{k=-\infty}^{\infty} \tilde{X}_1(e^{j(\omega-2\pi k)})$



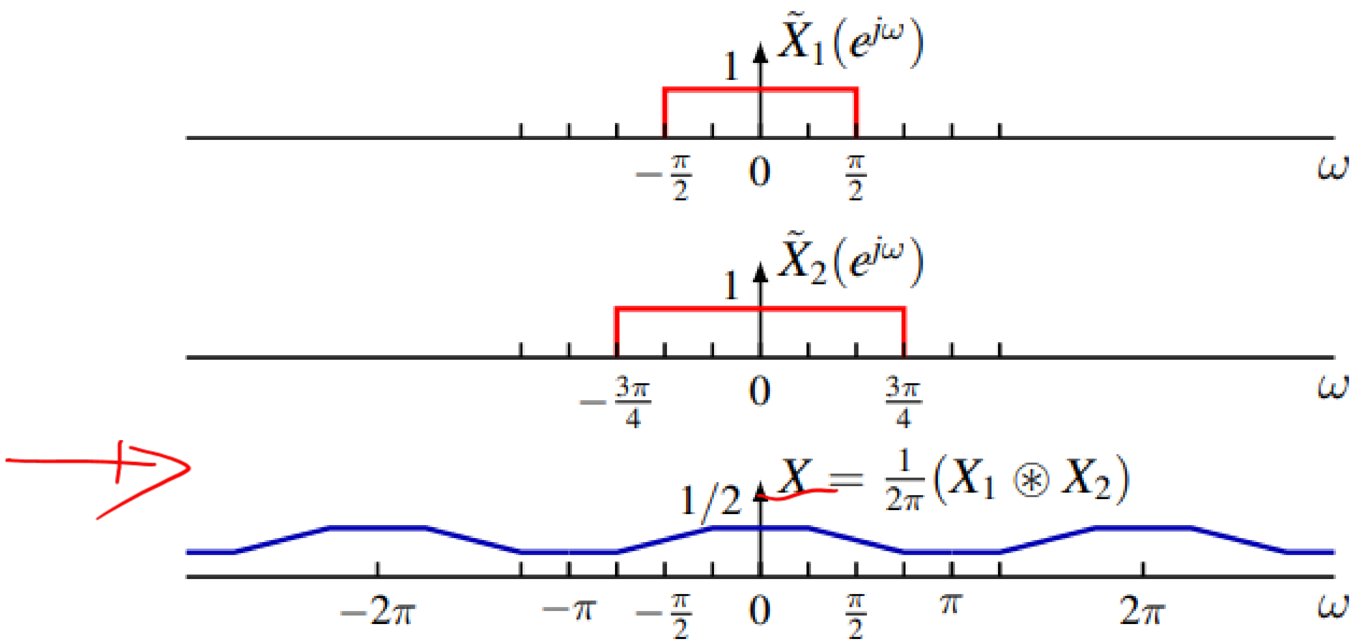
$\tilde{X}_2(e^{j\omega}) = \begin{cases} X_2(e^{j\omega}), & \omega \in (-\pi, \pi) \\ 0, & \text{otherwise} \end{cases}$

Example

- $x[n] = x_1[n]x_2[n]$, where $x_1[n] = \dfrac{\sin(3\pi n/4)}{\pi n}$, $x_2[n] = \dfrac{\sin(\pi n/2)}{\pi n}$

$$\frac{1}{2\pi}\left[X_1(e^{j\omega}) \circledast X_2(e^{j\omega})\right] = \frac{1}{2\pi}\left[\tilde{X}_1(e^{j\omega}) * X_2(e^{j\omega})\right] = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \tau_{2k\pi}\left[\tilde{X}_1(e^{j\omega}) * \tilde{X}_2(e^{j\omega})\right]$$

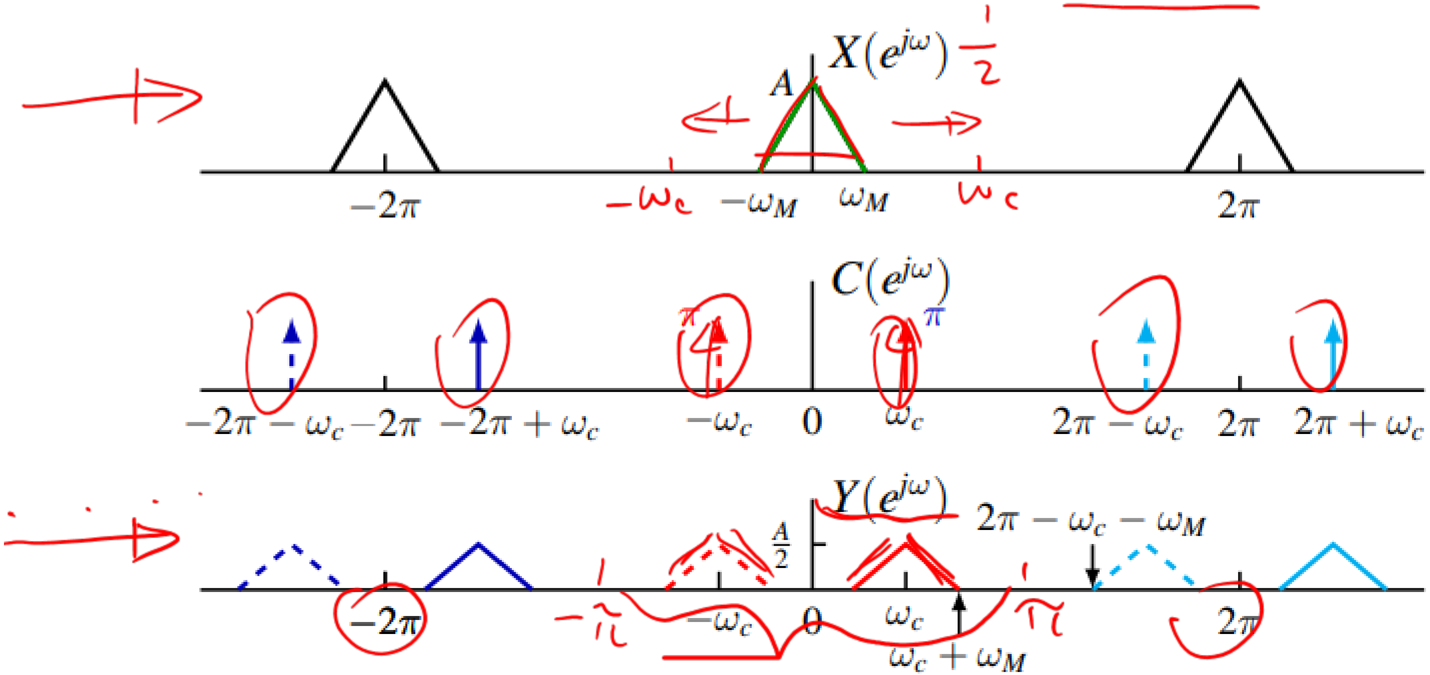
periodic — aperiodic — aperiodic



Example: DT Modulation

$\frac{1}{2}e^{j\omega_c n} + \frac{1}{2}e^{-j\omega_c n}$

$$y[n] = x[n]c[n], \text{where } c[n] = \cos(\omega_c n)$$



$\omega_c - \omega_M > 0$

- No overlap between replicas:

$$\begin{cases} \omega_c > \omega_M \\ \omega_c + \omega_M < \pi \end{cases} \Rightarrow \omega_M < \frac{\pi}{2}$$

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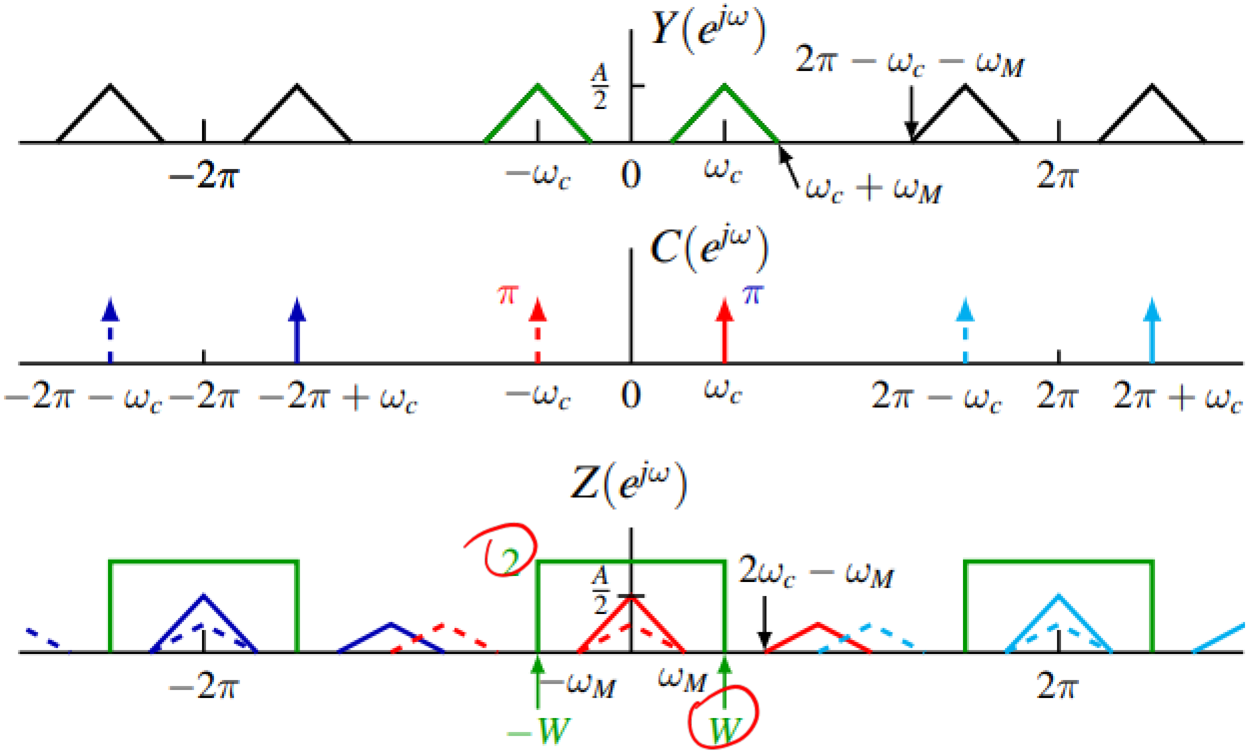
Example: DT Modulation

$$z[n] = y[n]c[n], \text{where } c[n] = \cos(\omega_c n)$$



- Recover $X(e^{j\omega})$ by lowpass filtering $Y(e^{j\omega})$ with $W \in (\omega_M, 2\omega_c - \omega_M)$

Duality in Fourier Analysis

- **CTFT:** Both **time and frequency** functions are continuous and aperiodic in general, **identical form except** for
  - different signs in exponent of complex exponential
  - constant factor $1/2\pi$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\, d\omega \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$$

- Suppose two functions are related by

$$f(r) = \int_{-\infty}^{\infty} g(\tau)e^{-jr\tau}\, d\tau$$

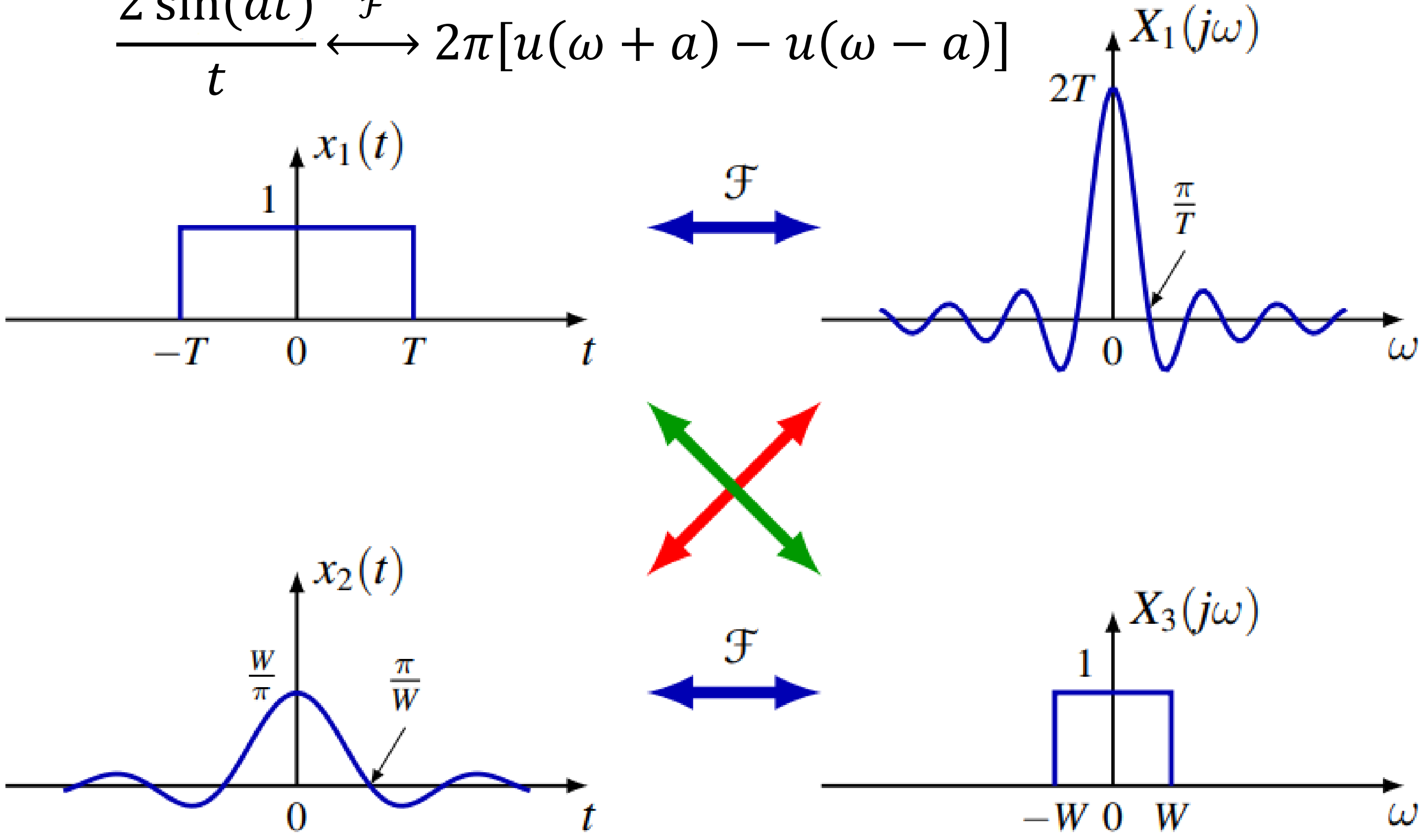
  - Let $\tau = t$, and $r = \omega$, $\Rightarrow\; g(t) \overset{\mathcal{F}}{\longleftrightarrow} f(\omega)$
  - Let $\tau = -\omega$, and $r = t$, $\Rightarrow f(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi g(-\omega)$

- **Duality:** $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) \;\Leftrightarrow\; X(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi x(-j\omega)$

Example of Duality in CTFT

$$u(t+a)-u(t-a) \overset{\mathcal{F}}{\longleftrightarrow} \frac{2\sin(a\omega)}{\omega}$$

$$\frac{2\sin(at)}{t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi[u(\omega+a)-u(\omega-a)]$$



Duality between DTFT and CTFS

DTFT pair
- discrete time
- continuous frequency

analysis equation

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = X(e^{j(\omega+2\pi)})$$

synthesis equation

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

CTFS pair
- continuous time
- discrete frequency

analysis equation

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt\,, \omega_0 = \frac{2\pi}{T}$$

synthesis equation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = x(t+T), \omega_0 = \frac{2\pi}{T}$$

continuous variable periodic functions $\xrightarrow{\text{CTFS}}$ doubly infinite sequences

doubly infinite sequences $\xrightarrow{\text{DTFT}}$ continuous variable periodic functions

LCCDEs

- Determine the **frequency response** of an LTI system described by

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- **Method 1:** using the **eigenfunction property**

let $$x[n] = e^{j\omega n} \rightarrow y[n] = H(e^{j\omega})e^{j\omega n}$$

substitution in to the difference equation yields

$$\sum_{k=0}^{N} a_k H(e^{j\omega})e^{j\omega(n-k)} = \sum_{k=0}^{M} b_k e^{j\omega(n-k)}$$

$$H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}$$

LCCDEs

- **Method 2:** taking the **Fourier transform** of both sides

$$\mathcal{F}\left\{\sum_{k=0}^{N} a_k y[n-k]\right\} = \mathcal{F}\left\{\sum_{k=0}^{M} b_k x[n-k]\right\}$$

- By **linearity** and **time shifting property**

$$\sum_{k=0}^{N} a_k e^{-j\omega k} Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-j\omega k} X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}$$

- $H(e^{j\omega})$ is a **rational function** of $e^{-j\omega}$, i.e., ratio of polynomials

Example

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$$

- **Frequency response**

$$H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}}$$

- Using **partial fraction expansion**

$$H(e^{j\omega}) = \frac{2}{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})} = \frac{4}{1 - \frac{1}{2}e^{-j\omega}} - \frac{2}{1 - \frac{1}{4}e^{-j\omega}}$$

- Taking inverse Fourier transform to find **impulse response**

$$h[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]$$

Example

$$y[n]-\frac{3}{4}y[n-1]+\frac{1}{8}y[n-2]=2x[n]$$

- Determine **zero-state response** to $x[n]=\left(\frac{1}{4}\right)^n u[n]$

$$Y(e^{j\omega})=H(e^{j\omega})X(e^{j\omega})=\frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}\cdot\frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

  - Using **partial fraction expansion**

$$Y(e^{j\omega})=-\frac{4}{1-\frac{1}{4}e^{-j\omega}}-\frac{2}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2}+\frac{8}{1-\frac{1}{2}e^{-j\omega}}$$

  - Taking inverse Fourier transform to find **output**

$$y[n]=\left\{-4\left(\frac{1}{4}\right)^n-2(n+1)\left(\frac{1}{4}\right)^n+8\left(\frac{1}{2}\right)^n\right\}u[n]$$

Q & A

Many Thanks